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The date of actual publication of any article, or portion of an article, may be found by referring to the first page of the sheet in which it is contained. The Author's private date is frequently given, at the end of an article.—ED.

ERRATA.

IN the memoir "On a Multiple Integral connected with the theory of Attractions," in the denominator of the value of U given by the formula (14), p. 221, *for* $\Gamma(\frac{1}{2}n-q)$ *read* $\Gamma(\frac{1}{2}n+q)$, and in the next line *for* $\Gamma(\frac{1}{2}n+q)$ *read* $\Gamma(\frac{1}{2}n-q)$.

THE

CAMBRIDGE AND DUBLIN

MATHEMATICAL JOURNAL.

ON THE ATTRACTION OF A SOLID OF REVOLUTION ON AN EXTERNAL POINT.

By George Boole.

I PROPOSE in this paper to determine the most general integral of which the differential equation

$$\frac{d^2u}{dx^2} + \frac{d^2u}{dy^2} + \frac{d^2u}{dz^2} = 0 \quad \ldots\ldots\ldots\ldots \quad (1)$$

is susceptible, when u is the potential of a solid of revolution on an external point, in such manner that the component attractions on that point are represented by $-\frac{du}{dx}, -\frac{du}{dy}, -\frac{du}{dz}$, respectively, and to consider the physical application of the result. This idea has been already applied in the case of the sphere, the integral being then a function of the distance of the attracted point from the centre; and Professor Challis has, I think, made a similar use of the equation, to determine, under certain circumstances, the motion of an incompressible fluid. But it has not, apparently, occurred to any one to apply a more general form of the integral except in Laplace's series. Allusion is sometimes made to a complete solution of the equation obtained by Poisson, but I have not been so fortunate as to meet with it. Certainly it does not appear to be involved in the solution of the well-known equation of elastic fluids. A form of the general integral which I have obtained, is too complex for physical applications, and is for the present reserved. But the case referred to, as the subject of this paper, admits of separate and comparatively easy discussion.

Let z be the axis of revolution of the solid, and r or $\sqrt{(x^2 + y^2)}$ the distance of the attracted point from that axis;

then u will be a function of z and r. The transformed equation is easily found to be

$$\frac{d^2u}{dr^2} + \frac{1}{r}\frac{du}{dr} + \frac{d^2u}{dz^2} = 0 \dots\dots\dots\dots (2),$$

which we proceed to discuss.

Writing the equation in the form

$$r^2\frac{d^2u}{dr^2} + r\frac{du}{dr} + r^2\frac{d^2u}{dz^2} = 0 \dots\dots\dots (3),$$

let $r = \varepsilon^\theta$, and let the symbol $\frac{d}{d\theta}$ be represented by D, then

$$r\frac{d}{dr} = D. \quad r^2\frac{d^2}{dr^2} = D(D-1);$$

$$\therefore \; r^2\frac{d^2u}{dr^2} + r\frac{du}{dr} = D^2u,$$

and the *symbolical* form of the differential equation is

$$D^2u + \frac{d^2}{dz^2}\varepsilon^{2\theta}u = 0 \dots\dots\dots\dots (4).$$

The method which we shall employ in the solution of this equation, is that developed in the *Philosophical Transactions* for 1844, Part II. (On a General Method in Analysis), and partially explained in the first Number of this *Journal* (On Laplace's Equation). We shall first obtain the complete integral by series. We shall then deduce a particular solution, in the form of a definite integral, and shall examine the relation which it bears to the different parts of the general solution.

The equation $D^2u = 0$ would give

$$u = A + B\theta \dots\dots\dots\dots\dots (5).$$

Substitute this value in (4); regarding A and B as variable parameters, we have

$$D^2A + \frac{d^2}{dz^2}\varepsilon^{2\theta}A + 2DB + \left(D^2B + \frac{d^2}{dz^2}\varepsilon^{2\theta}B\right)\theta = 0,$$

which affords the system of equations

$$D^2A + \frac{d^2}{dz^2}\varepsilon^{2\theta}A + 2DB = 0,$$

$$D^2B + \frac{d^2}{dz^2}\varepsilon^{2\theta}B = 0.$$

Of which the complete solution is

$$\left.\begin{array}{l} A = a_0 + a_2\varepsilon^{2\theta} + a_4\varepsilon^{4\theta} + \ldots \\ B = b_0 + b_2\varepsilon^{2\theta} + b_4\varepsilon^{4\theta} + \ldots \end{array}\right\} \ldots\ldots\ldots\ldots (6),$$

in which a_0 and b_0 are arbitrary functions of z, and the remaining coefficients are connected by the relations

$$m^2 a_m + \frac{d^2}{dz^2} a_{m-2} + 2mb_m = 0 \ldots\ldots\ldots (7),$$

$$m^2 b_m + \frac{d^2}{dz^2} b_{m-2} = 0 \ldots\ldots\ldots\ldots (8).$$

Hence, writing r for ε^θ, and $\log r$ for θ, we have

$$u = A + B \log r \ldots\ldots\ldots\ldots\ldots (9),$$

wherein

$$A = a_0 + a_2 r^2 + a_4 r^4 + \ldots$$
$$B = b_0 + b_2 r^2 + b_4 r^4 + \ldots$$

the relations (7) and (8) giving, as the law of derivation of the coefficients,

$$b_m = -\frac{1}{m^2}\frac{d^2}{dz^2} b_{m-2} \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (10),$$

$$a_m = -\frac{1}{m^2}\frac{d^2}{dz^2} a_{m-2} - \frac{2}{m}\frac{d^2}{dz^2} b_{m-2} \ldots\ldots\ldots\ldots (11).$$

This is the complete integral of the equation in series. We may remark that when $b_0 = 0$, all the succeeding values of b_m vanish, and the relation (11) gives

$$a_m = -\frac{1}{m^2}\frac{d^2}{dz^2} a_{m-2},$$

which is of the same form as (10). *Hence, if B vanishes, A assumes the general form of B.*

We will now deduce a particular solution in the form of a definite integral, and for this purpose, resuming (4), we have, on operating with the factor D^{-2},

$$u + \frac{1}{D^2}\frac{d^2}{dz^2}\varepsilon^{2\theta} u = 0 \ldots\ldots\ldots\ldots (12),$$

we retain no constants in the second member, because the equation which, after reduction, we shall actually integrate, will be of the second order, and will give us the proper number of arbitrary functions (*Phil. Trans.* p. 249).

Assume as the transformed equation

$$v + \frac{1}{D(D-1)}\frac{d^2}{dz^2}\varepsilon^{2\theta} v = 0 \ldots\ldots\ldots (13).$$

This equation is equivalent to $\frac{d^2u}{dr^2} + \frac{d^2u}{dz^2} = 0$, and gives

$$v = \phi(z + r\sqrt{-1}) + \psi(z - r\sqrt{-1}) \ldots\ldots (14).$$

Now (*Journal*, No. I. p. 13) we can transform the equation

$$u + \phi(D)\,\varepsilon^{2\theta}u = 0,$$

into

$$v + \psi(D)\,\varepsilon^{2\theta}v = 0.$$

if $\quad u = P_2 \frac{\phi(D)}{\psi(D)} v = \frac{\phi(D)\,\phi(D-2)\,\phi(D-4)\ldots}{\psi(D)\,\psi(D-2)\,\psi(D-4)\ldots} v \ldots (15).$

Hence, in the transformation above contemplated,

$$u = P_2 \frac{D-1}{D} v$$

$$= \frac{(D-1)(D-3)(D-5)\ldots}{D(D-2)(D-4)\ldots} v$$

$$= \frac{\left(\frac{D}{2} - \frac{1}{2}\right)\left(\frac{D}{2} - \frac{3}{2}\right)\left(\frac{D}{2} - \frac{5}{2}\right)\ldots}{\frac{D}{2}\left(\frac{D}{2} - 1\right)\left(\frac{D}{2} - 2\right)\ldots} v$$

$$= \frac{\Gamma\left(\frac{D}{2} + \frac{1}{2}\right)}{\Gamma\left(\frac{D}{2} + 1\right)} v$$

$$= \frac{1}{\Gamma(\frac{1}{2})} \frac{\Gamma(\frac{1}{2})\,\Gamma\left(\frac{D}{2} + \frac{1}{2}\right)}{\Gamma\left(\frac{D}{2} + 1\right)} v$$

$$= \frac{1}{\Gamma(\frac{1}{2})} \int_0^1 dt\,(1-t)^{-\frac{1}{2}}\, t^{\frac{D}{2} - \frac{1}{2}} v,$$

by a known theorem connecting the first and second of the Eulerian integrals.

Let $t^{\frac{1}{2}} = \sin w$; then, substituting

$$u = \frac{2}{\Gamma(\frac{1}{2})} \int_0^{\frac{\pi}{2}} dw\,(\sin w)^D v \ldots\ldots\ldots (16).$$

Now the general value of v is

$$v = \phi(z + r\sqrt{-1}) + \psi(z - r\sqrt{-1}),$$

but the series already obtained for u have no odd powers of r,

therefore v must have none, which condition can only be secured by the assumption

$$v = \phi(z + r\sqrt{-1}) + \phi(z - r\sqrt{-1}).$$

Substituting this value in (16), rejecting the needless factor $\frac{2}{\Gamma(\frac{1}{2})}$, and writing ε^θ for r, we have

$$u = \int_0^{\frac{\pi}{2}} dw\,(\sin w)^D \,\{\phi(z + \varepsilon^\theta\sqrt{-1}) + \phi(z - \varepsilon^\theta\sqrt{-1})\} \ldots . (17).$$

But $(\sin w)^{\frac{d}{d\theta}} f(\varepsilon^\theta) = f(\sin w\varepsilon^\theta)$, by Taylor's theorem,

$$\therefore \quad u = \int_0^{\frac{\pi}{2}} dw\,\{\phi(z + \sin w\varepsilon^\theta\sqrt{-1}) + \phi(z - \sin w\varepsilon^\theta\sqrt{-1})\}$$

$$= \int_0^{\frac{\pi}{2}} dw\,\{\phi(z + r\sin w\sqrt{-1}) + \phi(z - r\sin w\sqrt{-1})\}.$$

In the first term of the second member let $w = \frac{\pi}{2} - \theta$, we have

$$\int_0^{\frac{\pi}{2}} dw\,\phi(z + r\sin w\sqrt{-1}) = -\int_{\frac{\pi}{2}}^{0} d\theta\,\phi(z + r\cos\theta\sqrt{-1}).$$

$$= \int_0^{\frac{\pi}{2}} d\theta\,\phi(z + r\cos\theta\sqrt{-1}).$$

In the remaining term, let $w = \theta - \frac{\pi}{2}$, we have

$$\int_0^{\frac{\pi}{2}} dw\,\phi(z - r\sin w\sqrt{-1}) = \int_{\frac{\pi}{2}}^{\pi} d\theta\,\phi(z + r\cos\theta\sqrt{-1}).$$

Adding these, we have

$$u = \int_0^\pi d\theta\,\phi(z + r\cos\theta\sqrt{-1}) \ldots\ldots\ldots (18).$$

Now the complete integral is of the form $u = A + B\log(r)$; but the definite integral above found cannot involve such a factor as $\log(r)$. It must therefore represent that part of the complete integral, which remains when B vanishes. We have however seen that when B vanishes, A assumes the general form of B. Hence the definite integral is equivalent to B. We have therefore

$$u = A + \log(r)\int_0^\pi d\theta\,\phi(z + r\cos\theta\sqrt{-1}) \ldots . (19),$$

A being a series already determined, but reducible to a definite integral, viz.

$$A = \int_0^\pi d\theta\, \psi(z + r \cos \theta \sqrt{-1}) \ldots\ldots.(20),$$

whenever the definite integral in (19) vanishes.

Now if we suppose the second member of (19) to represent the potential of a solid of revolution on an external point, it is necessary that the definite integral in the second term should be assumed to vanish; for otherwise the value of u would be infinite, were the attracted point in the axis of revolution, since $r = 0$ renders $\log(r)$ infinite. We have therefore

$$u = A = \int_0^\pi d\theta\, \psi(z + r \cos \theta \sqrt{-1}) \text{ by } (20).$$

Let $f(z)$ represent the potential on any exterior point z, in the axis of revolution, then

$$f(z) = \int_0^\pi d\theta\, \psi(z) = \pi\psi(z).$$

$$\therefore \quad \psi(z) = \frac{1}{\pi} f(z),$$

whence $$u = \frac{1}{\pi} \int_0^\pi d\theta f(z + r \cos \theta \sqrt{-1}) \ldots\ldots (21),$$

which is the expression required.

Some interesting consequences flow from this theorem. If the potential of a solid of revolution on every external point in the axis be constant, we have

$$f(z) = c,$$

$$u = \frac{1}{\pi} \int_0^\pi c\, d\theta = c.$$

Hence the potential on points out of the axis will be constant also, and the attraction will vanish. We have examples of this case in some closed shells and hollow cylinders of infinite length, as respects points situate on their hollow interiors. Points exterior to the outer surface are not continuous with the above, and require a separate determination of the arbitrary function. But if the surface is not closed, however small may be the aperture, or if the cylinder is of finite length, all points within the concave or without the convex surface are to be considered as included in one application of the general formula.

It would be interesting to verify the general theorem of this paper, by applying it to the case of a circular ring, and comparing the result with the one obtained by ordinary integration. I shall simply indicate the equation, the truth of which would for this purpose require independent proof.

Let a be the diameter of the ring, the centre being the origin of coordinates. Then, ϕ representing an arc of the ring, the rest as before, the potential on the attracted point is easily found to be

$$u = \int_0^{2\pi} \frac{d\phi}{\sqrt{(a^2 - 2ar\cos\phi + r^2 + z^2)}} = 2\int_0^{\pi} \frac{d\phi}{\sqrt{(a^2 - 2ar\cos\phi + r^2 + z^2)}}.$$

Now the potential of the ring on the point z in the axis is $\frac{2\pi}{\sqrt{(a^2+z^2)}}$, therefore, by the general theorem,

$$u = \frac{1}{\pi}\int_0^{\pi} \frac{2\pi\, d\theta}{\sqrt{\{a^2 + (z + r\cos\theta\sqrt{-1})^2\}}}.$$

$$= 2\int_0^{\pi} \frac{d\theta}{\sqrt{\{a^2 + (z + r\cos\theta\sqrt{-1})^2\}}}.$$

Equating these expressions, we have

$$\int_0^{\pi} \frac{d\theta}{\sqrt{\{a^2 + (z + r\cos\theta\sqrt{-1})^2\}}} = \int_0^{\pi} \frac{d\phi}{\sqrt{\{a^2 - 2ar\cos\phi + r^2 + z^2)\}}} \ldots (22).$$

The discovery of relations like the above among definite integrals expressing in common the amount of some physical consequence, is not the least curious of the applications of the theorem.

Lincoln, Aug. 18, 1846.

ON A CERTAIN SYMBOLICAL EQUATION.

By George Boole.

In those preliminary researches on the Equation of Laplace's Functions, by which I was led to the method of solution exemplified in the first number of this *Journal*, a remarkable equation presented itself, which has appeared to me to be deserving of special and separate notice. This equation is a symbolical one, and it admits of two conjugate solutions, if the expression may be allowed, which are also purely symbolical; *i.e.* their validity does not depend on the significance of the symbols which they involve, but only on the truth of the laws of their combination. One interpretation of those symbols gives us Laplace's equation, but a more general interpretation than this is possible in the Integral Calculus, and there is perhaps, for I have not examined the question, an interpretation in the Calculus of Finite Differences. Practically, the solutions of Laplace's equation, which we are thus made acquainted with, are of little utility,

as compared with the one which I have already given; but they throw an interesting light on the subject of Symbolical Algebra, and serve to illustrate some general doctrines in Analysis.

The equation which we shall consider is the following, viz.

$$\pi_m \pi_n u + q\rho u = 0 \ldots\ldots\ldots\ldots\ldots (1),$$

in which u is the quantity to be determined, and the symbols π_m, π_n, ρ, applied to any subject u, combine according to the two laws

$$\pi_m \rho = \rho \pi_{m+1}, \quad \pi_n \rho = \rho \pi_{n+1} \ldots\ldots\ldots (2),$$

$$\pi_m \pi_n = \pi_n \pi_m + a(n-m)\rho \ldots\ldots\ldots (3).$$

The equations (2) are seen to be expressions of one law, π_m, π_n, differing only in the constants m and n. We suppose a to be an arbitrary constant.

Assume $u = \pi_{m+1} v$; we have by (1)

$$\pi_m \pi_n \pi_{m+1} v + q\rho\pi_{m+1} v = 0;$$

$$\therefore \quad \pi_m \pi_n \pi_{m+1} v + q\pi_m \rho v = 0 \quad \text{by (2)},$$

$$\pi_n \pi_{m+1} v + q\rho v = 0.^*$$

But $\pi_n \pi_{m+1} = \pi_{m+1}\pi_n + a(m+1-n)\rho$, by (3); therefore, on substitution,

$$\pi_{m+1}\pi_n v + \{q + a(m-n+1)\}\rho v = 0.$$

Let $v = \pi_{m+2} w$; then, by inspection,

$$\pi_{m+2}\pi_n w + \{q + a(m-n+1) + a(m-n+2)\}\rho w = 0.$$

Continuing these transformations, it is evident that, if we suppose in the original equation

$$u = \pi_{m+1}\pi_{m+2} \ldots \pi_{m+r} v,$$

we shall have

$$\pi_{m+r}\pi_n v + \{q + a(m-n+1) + a(m-n+2) \ldots + a(m-n+r)\}\rho v = 0.$$

Or $$\pi_{m+r}\pi_n v + \left\{q + ar(m-n) + a\frac{r(r+1)}{2}\right\}\rho v = 0.$$

If it then be possible by an integer value of r to satisfy the equation

$$q + ar(m-n) + a\frac{r(r+1)}{2} = 0 \ldots\ldots\ldots (4),$$

we shall have $$\pi_{m+r}\pi_n v = 0,$$

$$v = \pi^{-1}_n \pi^{-1}_{m+r} 0;$$

* Strictly $\pi_n \pi_{m+1} v + q\rho v = \pi^{-1}_m 0$. The assumption in the text is lawful, if the result (5) gives the requisite number of arbitrary constants; a condition which is satisfied in the example adduced.

$$\therefore\ u = \pi_{m+1}\pi_{m+2}\ldots\pi_{m+r}\pi_n^{-1}\pi^{-1}_{m+r}0\ldots\ldots\ldots(5),$$

which is a complete solution of the equation proposed.

As the equation determining r has two roots, it may be inferred that there are two solutions, which may be denominated as conjugate to each other. The existence and character of the second solution will be most distinctly presented by the following analysis.

Resuming the equation

$$\pi_m\pi_n u + q\rho u = 0.$$

Let us suppose $u = \pi_m^{-1}v$, then

$$\pi_m\pi_n\pi_m^{-1}v + q\rho\pi_m^{-1}v = 0$$

$$\{\pi_n\pi_m + a(n-m)\rho\}\,\pi_m^{-1}v + q\rho\pi_m^{-1}v = 0, \text{ by (3)},$$

$$\pi_n v + \{q + a(n-m)\}\,\rho\pi_m^{-1}v = 0,$$

$$\pi_{m-1}\pi_n v + \{q + a(n-m)\}\,\pi_{m-1}\rho\pi_m^{-1}v = 0.$$

But, by (2), $\pi_{m-1}\rho = \rho\pi_m$. Substituting,

$$\pi_{m-1}\pi_n v + \{q + a(n-m)\}\,\rho v = 0.$$

Hence it is evident that the compound substitution

$$u = \pi_m^{-1}\pi^{-1}_{m-1}\ldots\pi^{-1}_{m-r+1}v\ldots\ldots\ldots(6),$$

would give

$$\pi_{m-r}\pi_n v + \{q + a(n-m)\ldots + a(n-m+r-1)\}\,\rho v = 0\ldots(7).$$

Or
$$\pi_{m-r}\pi_n v + \left\{q + a(n-m)r + a\frac{r(r-1)}{2}\right\}\rho v = 0.$$

Hence if we determine r by the equation

$$q + a(n-m)r + a\frac{r(r-1)}{2} = 0\ldots\ldots(8),$$

we shall have

$$\pi_{m-r}\pi_n v = 0,$$

$$v = \pi_n^{-1}\pi^{-1}_{m-r}0,$$

$$u = \pi_m^{-1}\pi^{-1}_{m-1}\ldots\pi^{-1}_{m-r+1}\pi_n^{-1}\pi^{-1}_{m-r}0\ldots\ldots(9);$$

so that the two conjugate solutions, exhibited at one view, are

$$\left.\begin{aligned} u &= \pi_{m+1}\pi_{m+2}\ldots\pi_{m+r}\pi_n^{-1}\pi^{-1}_{m+r}0 \\ u &= \pi_m^{-1}\pi^{-1}_{m-1}\ldots\pi^{-1}_{m-r+1}\pi_n^{-1}\pi^{-1}_{m-r}0 \end{aligned}\right\}\ldots\ldots(10),$$

the values of r in the two cases being respectively determined by the equations

$$\left.\begin{aligned} q + a(m-n)r + a\frac{r(r+1)}{2} &= 0 \\ q + a(n-m)r + a\frac{r(r-1)}{2} &= 0 \end{aligned}\right\}\ldots(11).$$

The roots of the one equation are evidently those of the other with changed signs. If both solutions are available, each equation will have a positive and a negative root, the former belonging to the solution with which it is connected, the latter with its sign changed to the conjugate solution.

It is an obvious corollary from the above, that if α and β be constants, then the solution of the equation

$$(\pi_m + \alpha)(\pi_n + \beta)\, u + q\rho u = 0 \quad \ldots\ldots\ldots (12),$$

will be be exhibited in either of the conjugate forms,

$$\left.\begin{aligned} u &= (\pi_{m+1} + \alpha)(\pi_{m+2} + \alpha)\ldots(\pi_{m+r} + \alpha)\pi_n + \beta)^{-1}(\pi_{m+r} + \alpha)^{-1}\, 0 \\ u &= (\pi_m + \alpha)^{-1}(\pi_{m-1} + \alpha)^{-1}\ldots(\pi_{m-r+1} + \alpha)^{-1}(\pi_n + \beta)^{-1}(\pi_{m-r} + \beta)^{-1}\, 0 \end{aligned}\right\} \ldots (13),$$

the values of r being determined as before.

It remains to seek an interpretation of our symbols, and for this purpose let us assume

$$\pi_m = \phi(\mu)\frac{d}{d\mu} + m\phi'(\mu),\ \rho = \phi(\mu) \quad \ldots\ldots (14).$$

Then
$$\pi_m \rho u = \left\{\phi(\mu)\frac{d}{d\mu} + m\phi'(\mu)\right\}\phi(\mu)\, u,$$

$$= \phi(\mu)^2\frac{du}{d\mu} + \phi(\mu)\,\phi'(\mu)\, u + m\phi(\mu)\,\phi'(\mu)\, u,$$

$$= \phi(\mu)\left\{\phi(\mu)\frac{d}{d\mu} + (m+1)\,\phi'(\mu)\right\} u,$$

$$= \rho\pi_{m+1} u \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (15).$$

Secondly

$$\pi_m\pi_n u = \left\{\phi(\mu)\frac{d}{d\mu} + m\phi'(\mu)\right\}\left\{\phi(\mu)\frac{d}{d\mu} + n\phi'(\mu)\right\} u,$$

$$= \left\{\phi(\mu)\frac{d}{d\mu}\right\}^2 u + (m+n)\,\phi(\mu)\,\phi'(\mu)\frac{d}{d\mu}$$
$$+ n\phi(\mu)\,\phi''(\mu)\, u + mn\,\{\phi'(\mu)\}^2 u,$$

$$\pi_n\pi_m u = \left\{\phi(\mu)\frac{d}{d\mu}\right\}^2 u + (m+n)\,\phi(\mu)\,\phi'(\mu)\frac{du}{d\mu}$$
$$+ m\phi(m)\,\phi''(\mu)\, u + mn\,\{\phi'(\mu)\}^2 u:$$

therefore $(\pi_m\pi_n - \pi_n\pi_m)\, u = (n-m)\,\phi(\mu)\,\phi''(\mu)\, u,$

$$= (n-m)\,\rho\phi''(\mu)\, u.$$

Or, dropping the subject, u

$$\pi_m\pi_n = \pi_n\pi_m + (n-m)\,\rho\phi''(\mu)\, u \quad \ldots\ldots (16);$$

and that this may be identical with (3), we must have

$$\phi''(\mu) = a)$$

therefore $$\phi(\mu) = \frac{a}{2}\mu^2 + c_1\mu + c_2 \ldots\ldots\ldots\ldots (17),$$

c_1 and c_2 being arbitrary constants. Hence the laws of combination (2) and (3) are satisfied, if we assume

$$\pi_m = \left(\frac{a}{2}\mu^2 + c_1\mu + c_2\right)\frac{d}{d\mu} + m(a\mu + c_1), \rho = \frac{a}{2}\mu^2 + c_1\mu + c_2 \ldots (18);$$

and we are at liberty to substitute these values in the general equation (12), and in its conjugate solutions (13).

The equation of Laplace's Functions will be a particular case of the equation thus transformed. For, assume in (12) and (18),

$$m = 0, n = 0, a = -2, c_1 = 0, c_2 = 1, \alpha = \frac{d}{d\phi}\sqrt{-1}, \beta = -\frac{d}{d\phi}\sqrt{-1};$$

we have $$\pi_0 = (1-\mu^2)\frac{d}{d\mu}, \quad \rho = 1 - \mu^2,$$

$$\left\{(1-\mu^2)\frac{d}{d\mu} + \frac{d}{d\phi}\sqrt{-1}\right\}\left\{(1-\mu^2)\frac{d}{d\mu} - \frac{d}{d\phi}\sqrt{-1}\right\}u + q(1-\mu^2)u = 0.$$

Or $$(1-\mu^2)\frac{d}{d\mu}(1-\mu^2)\frac{du}{d\mu} + \frac{d^2u}{d\phi^2} + q(1-\mu^2)u = 0,$$

which, on assigning a proper value to q, is Laplace's equation, the solutions being

$$u = \left(\pi_1 + \frac{d}{d\phi}\sqrt{-1}\right)\left(\pi_2 + \frac{d}{d\phi}\sqrt{-1}\right)\ldots\left(\pi_r + \frac{d}{d\phi}\sqrt{-1}\right)$$
$$\left(\pi_0 - \frac{d}{d\phi}\sqrt{-1}\right)^{-1}\left(\pi_r + \frac{d}{d\phi}\sqrt{-1}\right)^{-1} 0 \ldots (19),$$

$$u = \left(\pi_0 + \frac{d}{d\phi}\sqrt{-1}\right)^{-1}\left(\pi_{-1} + \frac{d}{d\phi}\sqrt{-1}\right)^{-1}\ldots\left(\pi_{-r+1} + \frac{d}{d\phi}\sqrt{-1}\right)^{-1}$$
$$\left(\pi_0 - \frac{d}{d\phi}\sqrt{-1}\right)^{-1}\left(\pi_{-r} - \frac{d}{d\phi}\sqrt{-1}\right)^{-1} 0 \ldots (20);$$

where, in general, $\pi^\lambda = (1-\mu^2)\dfrac{d}{d\mu} + 2\lambda\mu$;

and the values of r are given respectively by the equations

$$q - r(r+1) = 0,$$
$$q - r(r-1) = 0.$$

If $q = n(n+1)$, as in our previous paper, then $r = n$ in the first solution, and $n + 1$ in the second.

We may remark that the process of reduction might have been so ordered as to have eliminated the last operating factor in each solution. This would have detracted from the generality of the first solution, in which there is but one other inverse factor, but not of the second. We have therefore for the sake of symmetry retained the factor in both. To shew how it might have been evaded, let us resume the equations (6) and (7), and writing the second in the form

$$\pi_n \pi_{m-r} v + \{q + a(n-m) \ldots + a(n-m+r-1) + a(n-m+r)\}\rho v = 0,$$

or $$\pi_n \pi_{m-r} v + \left\{q + a(n-m)(r+1) + a\frac{r(r+1)}{2}\right\} = 0,$$

we have $$\pi_n \pi_{m-r} v = 0, \quad \text{or } v = \pi_{m-r}^{-1} \pi_n^{-1} 0,$$

if $$q + a(n-m)(r+1) + a\frac{r(r+1)}{2} = 0.$$

Hence $u = \pi_m^{-1} \pi_{m-1}^{-1} \ldots \pi_{m-r+1}^{-1} \pi_{m-r}^{-1} \pi_n^{-1} 0$, or writing $r - 1$, for r

$$u = \pi_m^{-1} \pi_{m-1}^{-1} \ldots \pi_{m-r+1}^{-1} \pi_n^{-1} 0 \ldots\ldots\ldots (21),$$

if $$q + a(n-m)r + a\frac{r(r-1)}{2} = 0,$$

which differs only from the solution before obtained by the last factor.

The direct operations implied in the above solutions will involve differentiation, and the inverse ones the solution of a partial differential equation of the first order. We shall not exhibit the results, as it is clear that neither of the solutions can be freed from integral signs, but shall only remark that the second solution, freed as above from its last factor, is equivalent to the result obtained by Mr. Hargreave in the *Philosophical Transactions.*

The investigation we have entered upon is chiefly valuable, as presenting to us what will be thought a very curious chapter in symbolical algebra, and introducing us to the family of which Laplace's equation is a member. But it must be confessed that they are an interesting rather than an amiable group.

To give completeness to my former paper, I ought to have illustrated the general value of P_n deduced from the integral by actually calculating a few coefficients. This is an extremely simple matter, as all the operations are direct. It must be remembered that the product $1.2.\ldots.p$ like $\Gamma(p+1)$ becomes 1 when $p = 0$.

Lincoln, Aug. 18, 1846.

INVESTIGATION OF CERTAIN PROPERTIES OF THE ELLIPSOID.

By THOMAS WEDDLE, Newcastle-upon-Tyne.

*Conjugate points and diameters—Conjugate diametral and tangent planes—Conjugate parallelepipeds.**

Let
$$\left.\begin{array}{ll} l_1^2 + m_1^2 + n_1^2 = 1, & l_1 l_2 + m_1 m_2 + n_1 n_2 = 0 \\ l_2^2 + m_2^2 + n_2^2 = 1, & l_1 l_3 + m_1 m_3 + n_1 n_3 = 0 \\ l_3^2 + m_3^2 + n_3^2 = 1, & l_2 l_3 + m_2 m_3 + n_2 n_3 = 0 \end{array}\right\} \ldots\ldots (A).$$

Now these are the same relations as those that obtain among the directing cosines of three straight lines mutually at right angles, hence we must likewise have

$$\left.\begin{array}{ll} l_1^2 + l_2^2 + l_3^2 = 1, & l_1 m_1 + l_2 m_2 + l_3 m_3 = 0 \\ m_1^2 + m_2^2 + m_3^2 = 1, & l_1 n_1 + l_2 n_2 + l_3 n_3 = 0 \\ n_1^2 + n_2^2 + n_3^2 = 1, & m_1 n_1 + m_2 n_2 + m_3 n_3 = 0 \end{array}\right\} \ldots\ldots (B).$$

Also, from Lagrange's formulas, we get (Gregory's *Solid Geom.* p. 51),

$$\left.\begin{array}{lll} \pm l_1 = m_2 n_3 - m_3 n_2, & \pm l_2 = m_3 n_1 - m_1 n_3, & \pm l_3 = m_1 n_2 - m_2 n_1 \\ \pm m_1 = l_3 n_2 - l_2 n_3, & \pm m_2 = l_1 n_3 - l_3 n_1, & \pm m_3 = l_2 n_1 - l_1 n_2 \\ \pm n_1 = l_2 m_3 - l_3 m_2, & \pm n_2 = l_3 m_1 - l_1 m_3, & \pm n_3 = l_1 m_2 - l_2 m_1 \end{array}\right\}$$
$$\ldots\ldots (C).\dagger$$

Let
$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1 \ldots\ldots\ldots\ldots (1)$$

be the equation to an ellipsoid, and $(x_1 y_1 z_1)$, $(x_2 y_2 z_2)$, $(x_3 y_3 z_3)$ three conjugate points on it. The equations to the three conjugate tangent planes touching at these points will be

$$\frac{x_1 x}{a^2} + \frac{y_1 y}{b^2} + \frac{z_1 z}{c^2} = 1 \ldots\ldots\ldots\ldots (2),$$

$$\frac{x_2 x}{a^2} + \frac{y_2 y}{b^2} + \frac{z_2 z}{c^2} = 1 \ldots\ldots\ldots\ldots (3),$$

$$\frac{x_3 x}{a^2} + \frac{y_3 y}{b^2} + \frac{z_3 z}{c^2} = 1 \ldots\ldots\ldots\ldots (4).$$

---

* DEF. *Conjugate points* are the (three) extremities of conjugate diameters, and *conjugate tangent planes* touch the ellipsoid at conjugate points. Also a *conjugate parallelepiped* circumscribing an ellipsoid has its faces parallel to conjugate diametral planes. These terms are convenient, and they seem to be appropriate.

† [In all these formulæ the same sign must be used; the upper in the case when the two systems of axes (l_1, l_2, l_3), (l_1, m_1, n_1) are similarly arranged, and the lower sign when the arrangements are inverse, as, for instance, would be the case if one system were the image of the other in a mirror whose plane passes through the origin.]

The diameter conjugate to (2) is $\frac{x}{x_1} = \frac{y}{y_1} = \frac{z}{z_1}$, and as this line is the intersection of the diametral planes parallel to (3) and (4), we must have

$$\frac{x_1x_2}{a^2} + \frac{y_1y_2}{b^2} + \frac{z_1z_2}{c^2} = 0 \ldots\ldots\ldots\ldots (5),$$

and
$$\frac{x_1x_3}{a^2} + \frac{y_1y_3}{b^2} + \frac{z_1z_3}{c^2} = 0 \ldots\ldots\ldots\ldots (6).$$

By similarly considering the diameters conjugate to (3) or (4), we shall get one of the equations just deduced, together with the following,

$$\frac{x_2x_3}{a^2} + \frac{y_2y_3}{b^2} + \frac{z_2z_3}{c^2} = 0 \ldots\ldots\ldots\ldots (7).$$

Moreover the conjugate points being on the surface of the ellipsoid, we have

$$\frac{x_1^2}{a^2} + \frac{y_1^2}{b^2} + \frac{z_1^2}{c^2} = 1 \ldots\ldots\ldots\ldots (8),$$

$$\frac{x_2^2}{a^2} + \frac{y_2^2}{b^2} + \frac{z_2^2}{c^2} = 1 \ldots\ldots\ldots\ldots (9),$$

$$\frac{x_3^2}{a^2} + \frac{y_3^2}{b^2} + \frac{z_3^2}{c^2} = 1 \ldots\ldots\ldots\ldots (10).$$

Now, if in (A) we substitute $\frac{x_1}{a}, \frac{y_1}{b}, \frac{z_1}{c}, \frac{x_2}{a}$, &c. for l_1, m_1, n_1, l_2, &c., we shall get (8, 9, 10, 5, 6, 7), hence, as the results of this substitution are true, the equations (B) and (C) will still be true after undergoing the same transformation. We shall thus, after an obvious reduction, get the following equations,

$$x_1^2 + x_2^2 + x_3^2 = a^2 \ldots\ldots\ldots\ldots (11),$$

$$y_1^2 + y_2^2 + y_3^2 = b^2 \ldots\ldots\ldots\ldots (12),$$

$$z_1^2 + z_2^2 + z_3^2 = c^2 \ldots\ldots\ldots\ldots (13),$$

$$x_1y_1 + x_2y_2 + x_3y_3 = 0 \ldots\ldots\ldots\ldots (14),$$

$$x_1z_1 + x_2z_2 + x_3z_3 = 0 \ldots\ldots\ldots\ldots (15),$$

$$y_1z_1 + y_2z_2 + y_3z_3 = 0 \ldots\ldots\ldots\ldots (16).$$

$$\pm\frac{x_1}{a} = \frac{y_2z_3 - y_3z_2}{bc} \quad \pm\frac{x_2}{a} = \frac{y_3z_1 - y_1z_3}{bc} \quad \pm\frac{x_3}{a} = \frac{y_1z_2 - y_2z_1}{bc} \ldots (17).$$

$$\pm\frac{y_1}{b} = \frac{x_3z_2 - x_2z_3}{ac} \quad \pm\frac{y_2}{b} = \frac{x_1z_3 - x_3z_1}{ac} \quad \pm\frac{y_3}{b} = \frac{x_2z_1 - x_1z_2}{ac} \ldots (18).$$

$$\pm\frac{z_1}{c} = \frac{x_2y_3 - x_3y_2}{ab} \quad \pm\frac{z_2}{c} = \frac{x_3y_1 - x_1y_3}{ab} \quad \pm\frac{z_3}{c} = \frac{x_1y_2 - x_2y_1}{ab} \ldots (19).$$

Many of the preceding relations, (5)....(19), among the coordinates of three conjugate points are very neat; and some of them, so far as is known to me, have not been noticed before. They facilitate the investigation of several interesting properties of the ellipsoid, as will be shown below. It will be observed that these equations do not require the axes to be rectangular; they hold if the ellipsoid be referred to any system of conjugate diameters. In the following investigations, however, I shall suppose the axes to coincide with the principal diameters.

It may not be amiss to give the verbal statement of the geometrical properties implied in the three groups of equations (11, 12, 13), (14, 15, 16), and (17, 18, 19).

The first group (11, 12, 13) signifies that,

(A) If three conjugate points be projected on any diametral plane by lines drawn parallel to the diameter conjugate to this plane, the sum of the squares of the three lines of projection is equal to the square of the semidiameter.

The second group (14, 15, 16) shews that

(B) If from the points of projection mentioned in (A) lines be drawn parallel to, and be terminated by, any two conjugate diameters in the diametral plane, thus forming three parallelograms, the sum of two of these parallelograms is equal to the third.

Also from the third group (17, 18, 19), we have the following theorem:

(C) Let the parallelogram constructed on any two of three conjugate diameters, as well as the extremity of the third diameter, be projected, as in (A), on the plane of any two diameters of a second conjugate system. As the projection of the parallelogram is to the parallelogram constructed on the two diameters of the second system, so is the line which projects the extremity of the diameter to half the third diameter of the second system.

Let r_1, r_2, r_3 be the radii drawn to the three conjugate points, then will $r_1^2 = x_1^2 + y_1^2 + z_1^2$, &c.; wherefore, adding (11, 12, 13), we have

$$r_1^2 + r_2^2 + r_3^2 = a^2 + b^2 + c^2 \quad \ldots\ldots\ldots (20).$$

Hence

(D) The sum of the squares of any system of conjugate diameters is equal to the sum of the squares of the principal diameters.

Moreover (Gregory's *Solid Geom.* p. 17), the volume (V)

of a parallelepiped of which three contiguous edges meet at the origin and terminate in the three points $(x_1y_1z_1)$, $(x_2y_2z_2)$, $(x_3y_3z_3)$, is

$$V = (x_2y_3 - x_3y_2)\,z_1 + (x_3y_1 - x_1y_3)\,z_2 + (x_1y_2 - x_2y_1)\,z_3$$
$$= (19)\ \frac{ab}{c}\left\{z_1^{\,2} + z_2^{\,2} + z_3^{\,2}\right\} = (13)\ abc. \ldots\ldots\ldots (21).$$

Now the volume of the conjugate parallelepiped of which the conjugate tangent planes (2, 3, 4) are adjacent faces, is evidently eight times that just found; hence, we infer that

(E) Each conjugate parallelepiped circumscribing an ellipsoid, is equal to that constructed on the principal diameters.

Let $\widehat{r_1r_2}$ denote the angle included between the radii r_1 and r_2. The area of the parallelogram of which r_1 and r_2 are contiguous sides is $r_1r_2 \sin \widehat{r_1r_2}$, and the projections of this area on the planes of yz, zx, and xy, are $y_1z_2 - y_2z_1$, $x_2z_1 - x_1z_2$, and $x_1y_2 - x_2y_1$; hence, by the theory of projections (Gregory's *Solid Geom.* p. 14),

$$r_1^{\,2}r_2^{\,2} \sin^2 \widehat{r_1r_2} = (y_1z_2 - y_2z_1)^2 + (x_2z_1 - x_1z_2)^2 + (x_1y_2 - x_2y_1)^2;$$

or reducing, by means of (17, 18, 19),

$$\left.\begin{aligned} & r_1^{\,2}r_2^{\,2} \sin^2 \widehat{r_1r_2} = a^2b^2c^2 \left\{\frac{x_3^{\,2}}{a^4} + \frac{y_3^{\,2}}{b^4} + \frac{z_3^{\,2}}{c^4}\right\} \\ \text{Similarly}\quad & r_1^{\,2}r_3^{\,2} \sin^2 \widehat{r_1r_3} = a^2b^2c^2 \left\{\frac{x_2^{\,2}}{a^4} + \frac{y_2^{\,2}}{b^4} + \frac{z_2^{\,2}}{c^4}\right\} \\ \text{and}\quad & r_2^{\,2}r_3^{\,2} \sin^2 \widehat{r_2r_3} = a^2b^2c^2 \left\{\frac{x_1^{\,2}}{a^4} + \frac{y_1^{\,2}}{b^4} + \frac{z_1^{\,2}}{c^4}\right\} \end{aligned}\right\} \ldots (22).$$

Add these and reduce by (11, 12, 13),

$$r_1^{\,2}r_2^{\,2} \sin^2 \widehat{r_1r_2} + r_1^{\,2}r_3^{\,2} \sin^2 \widehat{r_1r_3} + r_2^{\,2}r_3^{\,2} \sin^2 \widehat{r_2r_3} = b^2c^2 + a^2c^2 + a^2b^2 \ldots (23),$$

which amounts to another well-known theorem; namely

(F) The sum of the squares of the parallelograms formed by each pair of conjugate diameters is equal to the sum of the squares of the rectangles under each pair of the principal diameters. Or, the sum of the squares of the faces of any conjugate parallelepiped is equal to the sum of the squares of the faces of the parallelepiped described on the principal diameters.

If p_1, p_2, and p_3 be the perpendiculars from the centre on the three conjugate tangent planes (2, 3, 4), we shall have

$$\left.\begin{aligned}\frac{1}{p_1^2} &= \frac{x_1^2}{a^4} + \frac{y_1^2}{b^4} + \frac{z_1^2}{c^4}\\ \frac{1}{p_2^2} &= \frac{x_2^2}{a^4} + \frac{y_2^2}{b^4} + \frac{z_2^2}{c^4}\\ \frac{1}{p_3^2} &= \frac{x_3^2}{a^4} + \frac{y_3^2}{b^4} + \frac{z_3^2}{c^4}\end{aligned}\right\} \dots\dots\dots \quad (24).$$

By addition, and (11, 12, 13),

$$\frac{1}{p_1^2} + \frac{1}{p_2^2} + \frac{1}{p_3^2} = \frac{1}{a^2} + \frac{1}{b^2} + \frac{1}{c^2} \dots\dots\dots (25).$$

Hence

(G) The sum of the squares of the reciprocals of the perpendiculars from the centre of an ellipsoid on three conjugate tangent planes is equal to the sum of the squares of the reciprocals of the principal semidiameters.

From (22) and (24), we have

$$r_1 r_2 \sin \widehat{r_1 r_2} = \frac{abc}{p_3}, \quad r_1 r_3 \sin \widehat{r_1 r_3} = \frac{abc}{p_2}, \quad r_2 r_3 \sin \widehat{r_2 r_3} = \frac{abc}{p_1} \dots (26).$$

We might, however, have deduced (23) otherwise. After having established (25) and (26), the latter of which is evidently only another form of (21), eliminate p_1, p_2, and p_3 between them, and we have (23) at once.

The locus of the intersections of three conjugate tangent planes will obviously be obtained by eliminating x_1, y_1, z_1, x_2, &c. from (2, 3, 4) by means of some of the succeeding equations. Now this elimination is immediately effected by taking the sum of the squares of (2, 3, 4) and reducing by (11)...(16),

$$\therefore \quad \frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 3 \dots\dots\dots\dots (27).$$

Hence

(H) The locus of the intersections of conjugate tangent planes to an ellipsoid is a concentric similar ellipsoid whose principal diameters are to those of the given ellipsoid as $\sqrt{3}:1$.

A different investigation of this may be seen in Gregory's *Solid Geom.* p. 269.

The theorem (H) may evidently be varied by saying

(I) Every conjugate parallelepiped circumscribing an ellipsoid is inscribed in a concentric similar ellipsoid.

We shall next establish the following theorem:

(K) If from the centre O of an ellipsoid any line be drawn intersecting the spheres described on the principal diameters $A'A$, $B'B$, $C'C$ in the points P, Q, R; and from P, Q, R

perpendiculars PX, QY, RZ be drawn to $A'A$, $B'B$, $C'C$ respectively; then, if a parallelepiped be described having OX, OY, and OZ for three contiguous edges, the corner S opposite to O will be a point in the ellipsoid.

For if l, m, n be the directing cosines of $OPQR$, we evidently have $OX = al$, $OY = bm$, and $OZ = cn$; and hence the point S or (al, bm, cn) is on the ellipsoid (1), since $l^2 + m^2 + n^2 = 1$.

$OPQR$ may be denominated the line *corresponding* to the point S, and conversely.

Putting the equation to the ellipsoid (1) under the form

$$x = al, \quad y = bm, \quad z = cn, \quad l^2 + m^2 + n^2 = 1 \;..(28),$$

it is evident from what we have just seen that the equation to the line corresponding to the point (al, bm, cn), is

$$\frac{x}{l} = \frac{y}{m} = \frac{z}{n} \;\ldots\ldots\ldots\ldots\ldots\; (29),$$

which is independent of a, b, c.

Hence if three conjugate points be denoted by (al_1, bm_1, cn_1), (al_2, bm_2, cn_2), and (al_3, bm_3, cn_3), the equations to the corresponding lines will be

$$\frac{x}{l_1} = \frac{y}{m_1} = \frac{z}{n_1} \;\ldots\ldots\ldots\ldots\ldots\; (30),$$

$$\frac{x}{l_2} = \frac{y}{m_2} = \frac{z}{n_2} \;\ldots\ldots\ldots\ldots\ldots\; (31),$$

$$\frac{x}{l_3} = \frac{y}{m_3} = \frac{z}{n_3} \;\ldots\ldots\ldots\ldots\ldots\; (32).$$

Also, since the three points are conjugate, we have (5, 6, 7),

$$\left.\begin{aligned} l_1l_2 + m_1m_2 + n_1n_2 &= 0 \\ l_1l_3 + m_1m_3 + n_1n_3 &= 0 \\ l_2l_3 + m_2m_3 + n_2n_3 &= 0 \end{aligned}\right\} \;\ldots\ldots\ldots\; (33).$$

Now (33) shews that the corresponding lines (30, 31, 32) are perpendicular to each other; also, it is evident that if the three lines be perpendicular to each other, the corresponding points will be conjugate, for in this case (33) is satisfied. Hence

(L) The three lines corresponding to three conjugate points on an ellipsoid are perpendicular to each other; and, conversely, if three lines be perpendicular to each other, the corresponding points are conjugate.

Hence, to find three conjugate points, three conjugate diameters, or three conjugate diametral planes, we may proceed thus:

(M) From the centre O of the ellipsoid draw any three straight lines at right angles to each other, and find the corresponding points L, M, N, by (K); then L, M, N are conjugate points; OL, OM, and ON conjugate semidiameters; and LOM, MON, and NOL conjugate diametral planes.

The preceding theorems are not offered as being principally new, though, so far as I know, several of them are so. I believe, however, that the investigations will be found to be original for the most part; these prove that the remarkable relations (5)...(19) may often be applied with great advantage in establishing many properties of the ellipsoid concerning conjugate diameters, &c., and to shew this is one of the objects I have kept steadily in view in writing this paper.

Newcastle-upon-Tyne, July 17, 1846.

ON PRINCIPAL AXES OF A BODY, THEIR MOMENTS OF INERTIA, AND DISTRIBUTION IN SPACE.

By Richard Townsend. F. T. C. D.

(*Continued from* Vol. I. p. 227.)

13. In general, equimomental axes, which are all equidistant from the centre of gravity of any body, group themselves on a number of cylinders enveloping all a sphere described round that centre; the axes of these cylinders passing all through the centre generate a central cone of equimomental axes, (the radii of the momental ellipsoid with which they coincide being therefore equal,) and their great circles of contact envelope therefore a sphero-conic, the intersection with the sphere of the cone *reciprocal** to that of the equimomental axes. Having the common moment of inertia, I, and the distance of the axes or the radius of the sphere, r, the equations of this conic are easily found.

For, denoting by I_1 the moment round the sides of the equimomental cone, we have $I_1 + Mr^2 = I$, and therefore $I_1 = I - Mr^2$. Substituting this value in (V), we get the

* Two cones of the second order having the same vertex, which are such that every tangent plane to either is perpendicular to a side of the other, have been called by M. Chasles, Supplementary, and by Prof. Mac Cullagh, Reciprocal Cones: the latter appellation perhaps expresses their connection better.

equation of that cone, from which, to get that of its reciprocal cone, we have but to change the three coefficients into their reciprocals, which will give

$$\frac{x^2}{A - I_1} + \frac{y^2}{B - I_1} + \frac{z^2}{C - I_1} = 0 \; \ldots\ldots\ldots \; (\alpha),$$

where I_1 has the value above; and this, with $r^2 = x^2 + y^2 + z^2$, will be the equations required.

This conic being a function of I and r, is therefore variable in magnitude, position, and figure, as these quantities vary, and will consequently follow in its successive changes of state no continuous law, if their variations be independent of each other; but if they be connected by any relation, the conic, depending then on but a single variable, will in passing through its several states describe a surface, which will obviously be of that extensive and very important class of central surfaces, of such frequent occurrence in physical investigations, those viz. whose intersections with every concentric sphere are all sphero-conics similarly placed, and which are contained all in the equation

$$\phi(a).x^2 + \chi(a).y^2 + \psi(a).z^2 = 0 \; \ldots\ldots\ldots \; (\beta);$$

where $a = x^2 + y^2 + z^2$, and when ϕ, χ, and ψ, are functions of any forms whatever, given, known, or to be determined, as the case may be: to this class belongs the wave surface of light in biaxal crystals, and it was from the present consideration that Professor Mac Cullagh obtained the symmetrical equation, and deduced many properties of that interesting surface. In the present case, to find the equation of the surface generated by the conics when the relation between I and r is given, we have but to substitute for I_1 in (α) its value in terms of r, (I being known in terms of the same from the given relation), and then in the result to change r into its value $\sqrt{(x^2 + y^2 + z^2)}$.

For instance, let the axes which generate the systems of enveloping cylinders for different spheres round the centre of gravity be all equimomental; then shall we have $I =$ a constant, and the equation will be

$$\frac{x^2}{x^2 + y^2 + z^2 - a^2} + \frac{y^2}{x^2 + y^2 + z^2 - b^2} + \frac{z^2}{x^2 + y^2 + z^2 - c^2} = 0 \ldots (\gamma),$$

where $\quad a^2, b^2, c^2 = \dfrac{I - A}{M}, \; \dfrac{I - B}{M}, \; \dfrac{I - C}{M}$, respectively.

As a second instance, let the axes be all *isochronal*, that is, such that a body would vibrate as a pendulum round them

all in the same time, then shall we have $\frac{I}{Mr}$ (= the radius of oscillation) = a constant = l; therefore $I_1 = M . r(l - r)$, and the equation will be

$$\frac{x^2}{a^2 - r(l-r)} + \frac{y^2}{b^2 - r(l-r)} + \frac{z^2}{c^2 - r(l-r)} = 0 \ldots\ldots (\delta),$$

where $a^2, b^2, c^2 = \frac{A}{M}, \frac{B}{M}, \frac{C}{M}$ = the squares of the three principal radii of gyration, and $r = \sqrt{(x^2 + y^2 + z^2)}$.

For a third instance, see next article.

14. Every system of equimomental axes, which are all equidistant from the centre of gravity, will obviously be also an isochronal system; whatever therefore has been said respecting such an equimomental system is true of such an isochronal system (13).

The direction of an equimomental system being given, the axes must lie all on a circular cylinder round the centre of gravity; but since every axis drawn at random in a body is isochronal with a parallel axis, in a plane passing through the centre of gravity, and distant from that point by an interval equal to its radius of oscillation, the rectangle under the distances of the two axes from the centre being equal to the square of their common radius of gyration; if the direction of an isochronal system be given, the axes will separate into two distinct systems, lying on two circular cylinders round the centre of gravity, whose radii for a given direction will be reciprocally proportional to each other, and which will therefore coincide only in the particular case when their common time of vibration is a minimum for that direction, in which case the radius of oscillation, or the length of the equivalent simple pendulum, is double the radius of gyration.

Taking, therefore, the ellipsoid reciprocal to the momental ellipsoid at the centre of gravity, the axes of absolute minimum time will describe a circular cylinder round the minimum axis of that ellipsoid with a radius equal to that semi-axis; and a cylinder similarly described round the maximum axis will be the locus of the axes of maximum minimorum times.

But for every intermediate value of the minimum time the axes will no longer lie on a single cylinder, but will group themselves on an infinite number of cylinders of equal radii, which will envelope all the same sphere round the centre of gravity, and whose axes passing all through that centre will generate a central equimomental cone.

The sides of this cone coinciding with a system of equal radii of the central ellipsoid, and every radius of that surface being the reciprocal of the coincident radius of gyration, the radius of the sphere enveloped by all the cylinders will be therefore equal to the reciprocal of the central radii which generate the equimomental cone, and the sphero-conic envelope of all their great circles of contact will be the intersection with that sphere of the reciprocal cone.

The equation of the surface generated by all these conics for the different systems of cylinders corresponding to different values of the minimum time of vibration may be immediately obtained.

For, in the equation (a) of the preceding article, we have I_1 connected with r by the relation $I_1 = Mr^2$; eliminating therefore that quantity, we get for the surface required the equation

$$\frac{x^2}{x^2+y^2+z^2-a^2} + \frac{y^2}{x^2+y^2+z^2-b^2} + \frac{z^2}{x^2+y^2+z^2-c^2} = 0 \ldots (\varepsilon);$$

where $a^2, b^2, c^2 = \frac{A}{M}, \frac{B}{M}, \frac{C}{M}$, a, b, c being the principal radii of gyration.

15. We may observe that the equation just determined is the same as Mr. Haughton, Fellow of Trinity College, Dublin, has found for the surface locus of the feet of central perpendiculars on all the tangent planes to the biaxal wave surface whose semi-axes are a, b, c. And that such should be the case may be readily shewn, for the surface at any point of a body, the squares of whose radii represent their movements of inertia, is the same as the surface of elasticity in the wave theory of light, (both surfaces will be in all respects similar at those points of a body where the three principal moments of inertia are proportional to the three principal elasticities of the medium); so that the surface round the centre of gravity, locus of the extremities of all the radii of gyration, is identical with the surface of elasticity in a medium whose three principal elasticities are proportional to the three principal central moments of inertia: the *apsidals,** therefore, of these two surfaces are also identical;

* Through a fixed point, taken arbitrarily in space, planes being drawn in all directions intersecting a given surface, and on a perpendicular to each erected at the same point, portions being taken equal to all the apsidal radii vectors of the curve of section, the surface locus of the extremities of all these portions is called the apsidal of the given surface. Professor Mac-

but of the former surface the apsidal is obviously (vide note *infra*) the surface whose equation has just been found; and of the latter, the apsidal is the locus of feet of perpendiculars on the wave surface, whose generating ellipsoid has for semi-axes those of the surface of elasticity.

Isochronal, like equimomental axes, if no otherwise restricted, are obviously infinite in number; like the latter, they are confined within certain limits in the neighbourbood of the centre of gravity; and as in the case of moments of inertia, a central surface may at every point of a body be readily found, whose radii shall represent the times of vibration round them*, and whose intersections with concentric spheres of different radii will give us the system of cones of isochronal axes for each point of the body; but this surface, not being of the second order, is little likely to prove interesting.

Before closing this article and proceeding to our more immediate subject, that of principal axes, it may be satisfactory to state to the reader that the properties we have been considering in this and in the two preceding articles are intimately connected with that subject, though at first sight they would seem to be altogether foreign to it. To shew that this is the case, we will here, in anticipation, state

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Cullagh, to whom the name is due and who was the first to consider this class of surfaces, has given the following method of constructing them and finding their equations. With its centre at the given point let a sphere of arbitrary radius be described intersecting the surface, then will every tangent plane to the cone which from the centre subtends the curve of intersection determine a section of the surface of which the side of contact will be an apsidal radius vector; the reciprocal cone will therefore intersect the same sphere in a curve belonging to the apsidal surface, so that to find the equation of that surface we have but to eliminate the parameter r, from the equation of the reciprocal cone, by means of that of the sphere $r^2 = x^2 + y^2 + z^2$. He has also shewn that every radius vector of the given surface, the corresponding radius of its apsidal, and the two perpendiculars on the tangent planes at the extremities of these radii, are respectively two and two, equal and at right angles to each other, and that all four lie always in the same plane, thus affording an obvious method of determining the tangent plane at any point of an apsidal when we know it for the corresponding point of the given surface; and shewing also that the apsidals (from the common pole) of two surfaces which are sphero-polar reciprocals to each other, are themselves also reciprocal surfaces. These results he has applied, in general, to the biaxal wave surface, which is the apsidal of an ellipsoid; and the last, in particular, to the two biaxal wave surfaces, apsidals to two reciprocal ellipsoids.

* In the same way as the radii of the momental ellipsoid at every point of a body represent the angular velocities with which the body would revolve round the coincident fixed axes, in consequence of impressed impulses having for all of them the same moment; which is obvious, as the angular velocities round the different axes will be then inversely as their moments of inertia, that is, directly as the squares of the radii.

a property of principal axes, which will be hereafter established and its consequences developed, but which will serve here to manifest the truth of the present assertion.

On every circular cylinder round an axis passing through the centre of gravity of any body there always exist two generatrices diametrically opposite to each other which enjoy the property of being principal axes, while in general no other generatrix of the same cylinder possesses that property; and these two particular generatrices are always those in which it is ultimately intersected by the consecutive *equimomental* cylinder which circumscribes the *same* sphere round the centre of gravity.

Assuming for the present the truth of this property, we see immediately that—

If to any sphere of arbitrary radius, described round the centre of gravity of a body, a system of tangent planes be drawn parallel to the system of planes tangent to any central *equimomental* cone, then will the developable surface envelope of that system of planes possess always the property, that its edges will be all principal axes. This theorem is due to Professor Thomson.

The curve of contact of every such developable surface with the sphere it envelopes being obviously a sphero-conic of the class we have been considering in the preceding articles, we hence see immediately a general property of that whole system of conics.—If each be made the curve of contact of a developable surface circumscribing the sphere on which it lies, then will that developable surface always possess the property that all its edges will be principal axes.

Taking now the whole system of developable surfaces thus related to the system of sphero-conics which generate the surface (ε), and observing that if a developable surface be at the same time circumscribed to a sphere and to any other surface whatever, then will its curve of contact with the former be a curve on the surface locus of feet of perpendiculars let fall from the centre of the sphere upon all the tangent planes to the latter, we get, remembering the opening remark of the present article, from the equation (ε) the following interesting property of principal axes.

In every body that particular system of axes which possess the twofold property of being axes of minimum time of vibration for their respective directions, and of being also principal axes, will always admit of an enveloping surface, and that envelope will be always the biaxal wave surface apsidal to the ellipsoid of gyration round the centre of gravity.

Again, for the same reason, from the equation (γ), which is exactly of the same form as (ϵ), and to which therefore the remark at the beginning of the present article equally applies, we get, by considering the system of developable surfaces similarly related to the system of conics which generate that surface, the property of principal axes discovered by Professor Thomson (see *Cambridge and Dublin Mathematical Journal*, vol. I. p. 203, Art. 21), a property for which he also (in a letter to the author) gave independently the present solution.

Every system of axes in a body which possess the twofold property of being both equimomental and principal will also envelope a biaxal surface round the centre of gravity, the apsidal of a surface of the second order concentric and coaxal with the ellipsoid of gyration round that point, and for different values of the common moment of inertia the whole system of apsidals envelopes of the different systems of equimomental principal axes will be always such that the generating surfaces of the second order, obviously all forming a concentric and coaxal system, will be all confocal, being (as is evident from the values of a, b, c there given) the system of surfaces of the second order conjugate to the surfaces of the system confocal with the ellipsoid of gyration, and therefore forming themselves also another confocal system, the well-known system conjugate to the former.

Professor Mac Cullagh, to whom indeed the whole theory of apsidals is due, was the first who considered geometrically the properties of a system of biaxal surfaces such as the present, the apsidals of a system of confocal surfaces of the second order. The author of the present paper hopes, in a subsequent article, in which the properties of the above system of envelopes will be exclusively considered, to be able to introduce to the reader's notice some remarkable and interesting properties of the important class of surfaces forming that system: but for the present we must discontinue a subject which was only introduced for the purpose of creating an interest in the properties discussed in the preceding article, and which, from its fundamental property not having been established but only assumed, could not but be considered as premature.

16. But to return to our more immediate subject, that of Principal Axes. From the important property of the centre of gravity with respect to parallel axes, it appears that we can find the moments of inertia round all axes assumed at pleasure in a body, if we know them for all axes through the

centre of gravity, or, which is the same thing, if we know the momental ellipsoid for that point. Hence, by means of that ellipsoid we may construct the momental ellipsoid for any other point we please of the body, and may therefore, by means of the same, find the principal axes at every point.

Now the equation of that ellipsoid contains six constants, which if known, we may consider the surface as determined: if, therefore, the body be terminated by any regular surface, and that it either be homogeneous, or that its density vary according to any regular law, we may assume arbitrarily three rectangular coordinate axes through the centre of gravity, and actually calculate for these the three moments of inertia and the three sums $\Sigma xydm$, $\Sigma zxdm$, $\Sigma yzdm$; or we may calculate the moments of inertia for any six axes making known angles with the same, and then equate the results to their known values in terms of the six constants, which will give us six linear equations to find these constants.

But as most bodies are altogether irregular, both in form and density, this method is seldom practicable, and we must have recourse to the experimental method, which holds in all cases and for all bodies.

By making the body (whatever be its nature) vibrate as a pendulum round any number of axes, assumed at pleasure, we can, from the observed time of a small oscillation round each axis of suspension, from the known distances of these axes from the centre of gravity, and from the known weight of the body, determine* the moments of inertia round the parallel axes through the centre of gravity; and these found for a sufficient number of axes will enable us to construct the momental ellipsoid for that point.

When therefore a body is given, we may consider as also given (since it may be always found) its momental ellipsoid at its centre of gravity, and consequently every curve or surface which is geometrically connected with that ellipsoid, or which may be derived therefrom.

We may therefore consider as given with every body the concentric ellipsoid (of which we have often made mention

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* By means of the well-known formula $T = \pi\sqrt{\frac{l}{g}}$, T being the time of vibration, and l the length of the equivalent simple pendulum; for, in this equation substituting for l its value $\frac{I + M.d^2}{M.d}$, I being the moment round the parallel central axis, we at once get $I = \frac{W.d.T^2}{\pi^2} - M.d^2$, $W = Mg$ being the weight of the body.

and which is of primary importance in every thing relating to the present subject) sphero-polar reciprocal to the momental ellipsoid round its centre of gravity, and every curve and surface geometrically connected therewith.

The momental ellipsoid round the centre of gravity being such that the squared reciprocals of its radii, multiplied each by the mass of the body, are equal to their moments of inertia, it follows that the reciprocal ellipsoid is such that the squares of the central perpendiculars on its tangent planes, multiplied each by the mass of the body, are equal to their moments of inertia; that ellipsoid round the centre of gravity of every body is therefore called the *ellipsoid of gyration*, and its semi-axes a, b, c, coinciding with the three central principal axes, are obviously the three principal radii of gyration.

At every point of a body, indeed, it is easy to see directly that the envelope of a plane, whose perpendicular distance therefrom is such that its square multiplied by the mass of the body is equal to the moment of inertia round that perpendicular, will be an ellipsoid, the reciprocal of the momental ellipsoid at the same point; for, let $a_1 b_1 c_1$ be the three particular distances which coincide with the principal axes, $A_1 B_1 C_1$ the three principal moments, and $\alpha_1 \beta_1 \gamma_1$ the direction angles of any other perpendicular p_1; then we shall have, moment of inertia round $p_1 = A_1 \cos^2 \alpha_1 + \beta_1 \cos^2 \beta_1 + C_1 \cos^2 \gamma_1$, that is, $M . p_1^2 = M . (a_1^2 \cos^2 \alpha_1 + b_1^2 \cos^2 \beta_1 + c_1^2 \cos^2 \gamma_1)$; the envelope therefore of the plane is the ellipsoid $a_1 b_1 c_1$.

We might have set out with establishing the existence of this latter ellipsoid at every point of a body, and by its aid have arrived at the results of the preceding articles; but by means of the momental ellipsoids they have been obtained perhaps more simply. Both ellipsoids, however, are important, and possess their advantages each over the other; for instance, in the problem of the rotation of a rigid body round a fixed point, which has been completely solved both by Poinsot and by Professor Mac Cullagh, the momental ellipsoid which has been employed by the former exhibits the whole motion with a clearness which could not be surpassed; and the reciprocal ellipsoid which has been used by the latter gives immediately, and without the trouble of any transformation, the elliptic integrals which express the circumstances of the motion, such as the times of oscillation, revolution, &c. Also in the more general case, when any external forces are acting, it exhibits more clearly the action of the different forces, centrifugal and external, in every position of the body.

In the particular case when the point is the centre of

gravity, the reciprocal ellipsoid is such that the central perpendiculars on its tangent planes are all equal to the coincident radii of gyration—that particular surface therefore (as above stated) is called the ellipsoid of gyration, and with every curve and surface derived from it, is to be considered as given with every given body. It is important in all that relates to the present subject, for by means of its varieties bodies may be conveniently classified, and we may always (as we shall presently see) suppose the remainder of a body, whatever be its nature, cut away, and confine our attention to that surface alone.

17. Having proved the existence at every point of a body of at least three principal axes mutually at right angles to each other, and having stated that there exists in every body an infinite number of points forming a curve for all of which the number of principal axes is infinite; we proceed now to the determination of their directions and of their moments of inertia at any particular point given or arbitrarily assumed in a body; to the consequent determination of that particular system of points which admit of an infinite number of principal axes; and subsequently, to the development of the general laws which govern the distribution in space of that class of lines in every body which enjoy the property of being principal axes.

Towards these objects we have the following theorem—

The dynamical principal axes at any point of a body are the geometrical principal axes of the cone which from that point as vertex envelopes the ellipsoid of gyration.

For the axes of that cone, which is the envelope of all the tangent planes drawn to the ellipsoid from the given point O, coincide with the axes of its reciprocal cone, and the axes of the latter cone, which is the locus of all the perpendiculars erected at the same point to those tangent planes, are the principal axes at that point; for all the sides of that cone are equimomental axes (6).

To shew this, let $\alpha\beta\gamma$ be the direction angles of the indefinite perpendicular P, erected at O to any tangent plane thence drawn to the ellipsoid, and let p be the length of the perpendicular upon that plane from the centre of gravity G, and q and r the distances of O from p and G respectively; then we shall have (P and p being parallel axes, distant by an interval q, and the latter passing through the centre of gravity,)

Moment of inertia round P = moment round $p + Mq^2$

$$= Mp^2 + Mq^2 = M(p^2 + q^2) = Mr^2 \ldots\ldots\ldots \text{(VI.)};$$

which being independent of the position of the tangent plane $(\alpha\beta\gamma)$ shews that the perpendiculars, that is the sides of the reciprocal cone, are all equimomental axes.

The construction thus indicated for determining the principal axes at every point of a body, enables us to conceive their directions at each point, and their relative positions at different points, as easily as could perhaps be desired. It also confirms the anticipations of Art. 11, respecting the symmetrical distribution of principal axes in the eight regions of space determined by the three principal planes through the centre of gravity.

18. By using the known property, that the three axes of the cone real or imaginary which from any vertex envelopes an ellipsoid, are the normals to the three surfaces of the second order confocal with the ellipsoid which pass through that vertex, we may substitute for the above construction another, virtually the same, but affording the advantage of enabling us to apply to principal axes in general the known properties of those lines which are normals to a system of confocal surfaces of the second order. Hence,

The principal axes at any point of a body are the normals to the three surfaces of the second order confocal with the ellipsoid of gyration which pass through that point.

This latter construction has been directly established by Professor Mac Cullagh from the dynamical property of principal axes, as follows:

Let O be any point of a body, G its centre of gravity, OX, OY, OZ the normals to the three surfaces of the second order confocal with the ellipsoid of gyration (a, b, c) which pass through O, and GP, GQ, GR the three central perpendiculars on the corresponding tangent planes at O to these three surfaces (all these lines being supposed produced indefinitely). He has shewn, that if the body revolve round either of the three normals (OX), then will the centrifugal forces of all its elements compound a single resultant passing through O.

ω being the angular velocity of rotation, let O' be the point on the ellipsoid of gyration where the normal $O'X'$ is parallel to OX, and P' the point where the tangent plane at O' meets (at right angles) the perpendicular GP; then, from a known property of confocal surfaces, will the area of the right-angled triangle $GO'P'$ be equal to that of the right-angled triangle GOP, and both triangles will lie in the same plane containing the three parallel lines GP, OX, $O'X'$.

dm being an element of the body at any point *o*, let *r* and *q* be the points where a plane through *o* perpendicular to the parallel axes *OX*, *GP* meets these axes respectively, and let *op* be a line drawn in that plane from *o* parallel and equal to the distance *OP*; the centrifugal force of *dm*, $\omega^2.or.dm$, which acts from *r* to *o* along the line *ro*, may be resolved into two; $\omega^2.oq.dm$ passing through the axis *GP*, and directed perpendicular to that axis from *q* to *o*; and $\omega^2.op.dm$ acting parallel to the line *OP*, in the direction from *O* to *P*.

A similar resolution being effected for every point of the body, we shall have the whole system of centrifugal forces arising from the rotation round *OX* replaced by two different and distinct systems of new forces; a system of parallel forces, $\omega^2.OP.dm$, acting all in the same direction from *O* to *P*, which are the same as if all the elements of the body were acted upon by equal and parallel accelerating forces, equal each to the quantity $\omega^2.OP$, parallel all to the line *OP*, and acting all from *O* to *P*; and a system of forces, $\omega^2.oq.dm$, passing all through the axis *GP*, and acting perpendicularly out from that axis, which are precisely the same in magnitude and direction as would arise from a rotation with the same angular velocity, ω, round *GP* as axis; the former of these systems will compound a single resultant, passing through the centre of gravity, parallel to the line *OP*, acting from *O* to *P*, and equal to the quantity $\omega^2.M.OP$; and the latter system, transferred all to the centre of gravity, will be again replaced by a system of forces passing all through that point, and by a system of moments in planes passing all through *GP*, the system of transferred forces will all equilibrate round the centre (2), and the system of moments will compound a resultant moment passing through the axis *GP*, the same in magnitude, plane, and direction as would result from a rotation round that axis with the angular velocity ω.

Now the ellipsoid of gyration being the reciprocal of the momental ellipsoid at the centre of gravity, the central perpendicular *p*, on the tangent plane at any point of either, and the radius *r*, drawn to the point of contact, coincide respectively with, and are the reciprocals of, the radius *r'*, drawn to the corresponding point of the other, and the perpendicular *p'*, on the tangent plane at the same. Hence, from (4) it appears that, If a body revolve round the central perpendicular (*p*) on any tangent plane to its ellipsoid of gyration, the plane of the resultant centrifugal moment, passing through that perpendicular, will be that of the radius (*r*) and perpendicular, the direction of that moment will be from the radius

towards the perpendicular, and its magnitude will be $\omega^2.M.$ (2 area of right-angled triangle rp).

The whole system of centrifugal forces arising from the rotation round OX, may therefore be finally replaced by a single force $\omega^2.M.OP$, passing through the centre of gravity, and acting in the direction from O' towards P', and by a moment $\omega^2.M.GP'.O'P'$, in the plane $GO'P'$, and acting in the direction from O' to P'; but, the plane $GO'P'$ coinciding with the plane GOP, and the rectangle $GP'.O'P'$ being equal to the rectangle $GP.OP$, the moment is equivalent to a moment in the plane GOP, equal to $\omega^2.M.GP.OP$, and acting from O to P, that is, to two equal and parallel forces $\omega^2.M.OP$, one passing through O and acting from O to P, and the other passing through the centre of gravity and acting in the opposite direction; the latter of these forces will destroy the single force to which it is equal and directly opposed, and we shall have remaining, as the final equivalent to the whole system of centrifugal forces arising from the rotation round OX, but a single force passing through O.*

Hence the normal at any point of a surface of the second order described in a body confocal with its ellipsoid of gyration is a principal axis at that point. And a similar proof holding for the other normals shews that the principal axes at any point of a body are the normals to the three surfaces of the second order confocal with its ellipsoid of gyration which pass through that point.

19. If the cone, real or imaginary, which from O as vertex envelopes the ellipsoid, be of revolution, the first of these constructions (17) shews that such a point admits of an infinite number of principal axes in the plane perpendicular to the internal axis of the cone; but if it be not of revolution, the point will admit of only three.

In general therefore (A, B, C being all unequal) the great majority of points in a body admit of but three principal axes; for when the three principal central moments are all unequal, the cone which envelopes the ellipsoid of gyration is generally not of revolution.

But for every surface of the second order there exist two different and distinct systems, real or imaginary, of enveloping cones of revolution, and the loci of their vertices are the two real focal conics of the surface.

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* The principles involved in the above demonstration were investigated by Professor Mac Cullagh for a different purpose: he merely applied them incidentally to the present question.

Hence, in every body there exist two continuous and distinct series of points which admit of an infinite number of principal axes (5), and the loci of these two series are both plane curves, of the second order, lying each in a principal plane through the centre of gravity, one an ellipse in the plane of AB, the other an hyperbola in the plane of AC, the real focal conics viz. of the ellipsoid of gyration a, b, c.

At every point of these two curves the normal plane is that which contains the infinite number of principal axes; for the tangent at any point on either of its focal conics is the internal axis of the cone of revolution which from that point as vertex envelopes any surface of the second order.

Lest imaginary cones might be here considered as a difficulty, it may be well to shew that the other construction (18) leads readily to the same results.

For, the focal conics common to a confocal system of surfaces of the second order, bound portions of the principal planes in which they lie, which are the infinitely flat surfaces of the system, and the transition state from one species to another; all planes, therefore, which pass through a tangent to a focal conic are tangent planes to these particular surfaces, and the perpendiculars to these planes at the point of contact which there generate the normal plane to that curve, are all normals to the same.

Two, therefore, of the three surfaces confocal with the ellipsoid abc, which pass through any point on either of its focal conics, admit of an infinite number of normals, lying all in the normal plane to the conic at that point; the body, therefore, at such a point admits of an infinite number of principal axes in that plane. (This is the general property of which particular cases were established on other principles in 12.)

20. It was proved above (17) that the cone reciprocal to that which from any point O envelopes the ellipsoid of gyration abc, is a cone of equimomental axes for that point; more generally, the whole system of cones reciprocal to those which from O as their common vertex envelope the whole system of surfaces confocal with that ellipsoid, will be the system of equimomental cones for that point (6).

For, every equimomental system of cones will be all coaxal, and will have all the same cyclic planes (6), its reciprocal system of cones will be therefore all coaxal, and will have all the same focal lines; but this is the known property of the system of cones which from any vertex envelope a system of confocal surfaces of the second order.

The same may be proved directly, which will thus, conversely, establish the more difficult property of confocal surfaces.

For, let a', b', c' be the semiaxes of any surface confocal with abc, then shall we have $a'^2 = a^2 + \delta$, $b'^2 = b^2 + \delta$, $c'^2 = c^2 + \delta$ where δ may have any value from $+\infty$ to $-a^2$); then, denoting by α', β', γ' the direction angles of P' the perpendicular erected at O to any tangent plane thence drawn to this surface, by p' the perpendicular from the centre on that tangent plane, by q' the distance between P' and p', and by r (as before) the distance of O from the centre, we shall have (p' and P' being parallel, the former passing through G),

$$\text{moment of inertia round } P' = \text{moment round } p' + Mq'^2$$
$$= M.(a^2\cos^2\alpha' + b^2\cos^2\beta' + c^2\cos^2\gamma') + M.q'^2$$
$$= M(p'^2 - \delta) + M.q'^2 = M(p'^2 + q'^2) - M.\delta$$
$$= M.r^2 - M.(a'^2 - a^2) \ldots\ldots\ldots\ldots\ldots\ldots\ldots (\text{VII.});$$

which, being a quantity independent of the position of the tangent plane, shews that P' generates a cone of equimomental axes. The system of cones generated by the perpendiculars are therefore all equimomentatl, consequently (6) they are all coaxal and concyclic; the enveloping system of cones, their reciprocals, are therefore all coaxal and confocal.

(From the above theorem, of which that in (17) is a particular case, we may, if in constructing for principal axes by means of the latter at a point *within* the ellipsoid of gyration the imaginary enveloping cone prove a source of difficulty, substitute for that ellipsoid any other confocal with it to which the point shall be external, and thus get a real coaxal cone.)

21. In the cases of the three particular confocal surfaces of the system which pass through O, the enveloping cones degenerate into three planes, the tangent planes to the surfaces, and their reciprocals therefore into the three normals; these normals are therefore the three common axes of all the enveloping cones, that is, the principal axes at O (18).

By substituting, therefore, in the known expressions which give the directions of the normal at any point of a surface of the second order in terms of the coordinates of the point and the semi-axes of the surface, the three values of δ which correspond to these three surfaces, we shall have the direction cosines of the principal axes at any point O expressed analytically in terms of known quantities.

Again, by putting into the quantity $M(r^2 - \delta)$, (which VII. expresses the value of the moment of inertia common to the

cone of perpendiculars whose parameter is δ,) the same three values of δ, we shall have the moments of inertia round the normals to the three surfaces, that is, the three principal moments at O.

Having got, indeed, the directions of the principal axes at any given point, we get at once the principal moments also from the known distances and moments of the parallel axes through the centre of gravity; but the above has the advantage of giving the values of the moments without requiring that the directions of the axes be previously known.

The three values of δ, which thus give the principal axes and the principal moments at any given point O, are the roots of a cubic equation; for, let xyz be the coordinates of O parallel to abc, and $a_{,}a_{,,}a_{,,,}$ the primary semi-axes of the three confocals, where $a_{,}^2 = a^2 + \delta_{,}$, $a_{,,}^2 = a^2 + \delta_{,,}$, $a_{,,,}^2 = a^2 + \delta_{,,,}$, then are these three values of δ obviously got from the equation

$$\frac{x^2}{a^2+\delta} + \frac{y^2}{b^2+\delta} + \frac{z^2}{c^2+\delta} - 1 = 0 \ldots\ldots \text{(VIII.)}$$

A cubic in which the successive substitutions for δ in the left-hand member of the quantities $+\infty$, $-c^2$, $-b^2$, and $-a^2$, give results with the respective signs $- + - +$, and which has, therefore, three real roots lying between these limits, as indeed from the nature of moments of inertia it could not but have.

Denoting now by $I_{,}$, $I_{,,}$, $I_{,,,}$ the three principal moments at O, we have

$$\left.\begin{aligned} I_{,} &= M\,.\,(r^2-\delta_{,}) = M\,.\,r^2 - M\,.\,(a_{,}^2-a^2) \\ I_{,,} &= M\,.\,(r^2-\delta_{,,}) = M\,.\,r^2 - M\,.\,(a_{,,}^2-a^2) \\ I_{,,,} &= M\,.\,(r^2-\delta_{,,,}) = M\,.\,r^2 - M\,.\,(a_{,,,}^2-a^2) \end{aligned}\right\} \ldots \text{(IX.)},$$

which we therefore know when we can find $a_{,}$, $a_{,,}$, $a_{,,,}$.

22. If the point were such that two of its principal moments of inertia were equal, then should two of these quantities $a_{,}$, $a_{,,}$, $a_{,,,}$ be equal, and this is the case for every point on either of the focal conics of abc. Hence the result obtained before (19), that in every body there exists two curves, loci of points, that admit of an infinite number of principal axes, viz. the focal conics of the ellipsoid of gyration.

If the point were such that the three principal moments were equal, then should $a_{,}$, $a_{,,}$, $a_{,,,}$, be all three equal, or the point should be upon both focal conics; but this in general would be impossible, for the focal conics of an ellipsoid have never a point in common, except in the particular case of a *prolate*

spheroid, where they both meet at the two foci. In the majority of bodies, therefore, there exists no point for which all axes are principal, and even in the limited class to which such points are confined (those, viz. for which two of the three central principal moments are equal and both less than the third) there never exist more than two; these are always on the central principal axis of unequal moment, and at opposite sides of and equidistant from the center of gravity, and are the foci of the ellipsoid of gyration which is then a prolate spheroid, which results were obtained before on other principles in (12).

That in such bodies all axes are principal at the two foci of its ellipsoid of gyration appears also from the general construction for principal axes at any point. For the two particular surfaces confocal with a prolate spheroid which pass through its foci, are both infinitely slender surfaces of revolution, one a prolate spheroid and the other an hyperboloid of two sheets, and the summits of these two surfaces meet at the foci; all planes, therefore, passing through these points are tangent planes to these two particular surfaces, and consequently all lines which pass through either point of cantact are there normals to the same: for both points, therefore, all axes are principal.

Their common distance on the axis of unequal moment A is easily obtained in terms of the two moments A and B; for calling it c, we have $c^2 = a^2 - b^2 = \dfrac{A - B}{M}$, and therefore $c = \pm\sqrt{\left(\dfrac{A - B}{M}\right)}$, its well known value (5).

In the limiting case, when the ellipsoid of gyration is a sphere, that is, in bodies for which AB and C are all three equal, there will obviously be but one such point, the centre of gravity itself.

23. The equation which contains the whole system of equimomental cones at any given point O referred to their principal axes may be readily obtained in terms of $a_{,}$, $a_{,,}$, $a_{,,,}$, the primary semi-axes of the three confocals which pass therethrough.

For, let a_0 be the primary semi-axes of any fourth surface confocal with abc, and I_0 the moment of inertia common to all the sides of the cone reciprocal to that which from O envelopes that surface, then (6) will the equation of that cone referred to its principal axes be

$$(I_{,} - I_0)\,\xi^2 + (I_{,,} - I_0)\,\eta^2 + (I_{,,,} - I_0)\,\zeta^2 = 0,$$

ξ, η, and ζ being the running coordinates parallel to those axes.

Now, $I_{,} = M \cdot r^2 - M \cdot (a^2_{,} - a^2)$ and $I_0 = M \cdot r^2 - M \cdot (a^2_0 - a^2)$, therefore $I_{,} - I_0 = M \cdot (a_{,}^2 - a_0^2)$; and similarly, we get $I_{,,} - I_0 = M \cdot (a^2_{,,} - a^2_0)$, and $I_{,,,} - I_0 = M \cdot (a^2_{,,,} - a^2_0)$.

Putting these values into the above, we have for the equation of the system of equimomental cones at 0,

$$(a_{,}^2 - a_0^2)\,\xi^2 + (a^2_{,,} - a_0^2)\,\eta^2 + (a^2_{,,,} - a_0^2)\,\zeta^2 = 0 \ldots\ldots (\text{X.}),$$

a_0 being the variable parameter.

The system of cones contained in this equation being reciprocal to those which from the same vertex envelope the system of surfaces confocal to abc, we can get immediately from the equation of the former system that of the latter, the direct investigation of which is by no means easy. For we have merely to change the coefficients of the variables into their reciprocals, which in the present instance will give

$$\frac{\xi^2}{a_{,}^2 - a_0^2} + \frac{\eta^2}{a_{,,}^2 - a_0^2} + \frac{\zeta^2}{a^2_{,,,} - a_0^2} = 0 \ldots\ldots\ldots (\text{XI.})$$

for the equation referred to its axes of the cone which from any given vertex envelopes the ellipsoid a_0, b_0, c_0; or by giving different values to a_0, of the system of cones which from the same vertex envelope the system of surfaces confocal with a_0, b_0, c_0, or with abc.

This equation (investigated directly) is due to Mr. Salmon, Fellow of Trinity College, Dublin, and also to Professor Mac Cullagh, who obtained it independently. It has been most successfully employed by the latter in the difficult problem of ellipsoidal attraction on an external point; and it is extensively important in the analytical theory of confocal surfaces, as shewing immediately that for the same vertex the cones which envelope such a system of surfaces are all coaxal and confocal.*

If the point O be such that the system of cones (X. or XI.) are all of revolution, then must some two of the quantities $a_{,}$, $a_{,,}$, $a_{,,,}$ be equal; this gives us the result obtained before, that the loci of the vertices of such a system of cones are the local conics of the ellipsoid of gyration.

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* Stating that well known theorem as follows: If a cone envelope a surface of the second order, then will every cone confocal with the former envelope a surface confocal with the latter. The author of the present paper has given the following extension of it: If any surface of the second order envelope another, then will every surface confocal with either envelope a surface confocal with the other.

If O be the centre of gravity, then will the cones enveloping the system of ellipsoids be all imaginary, while those which envelope the system of hyperboloids of both kinds will become the asymptotic cones of that system of surfaces. Hence in every body the system of cones reciprocal to the system of equimomental cones at its centre of gravity, will be the asymptotic cones to the system of hyperboloids confocal with its ellipsoid of gyration.

Substituting in (XI.) the values of $a_{\prime}^2$, $a_{\prime\prime}^2$, $a_{\prime\prime\prime}^2$, which correspond to this particular point (viz. $a_{\prime}^2 = a^2 - c^2$, $a_{\prime\prime}^2 = a^2 - b^2$, and $a_{\prime\prime\prime}^2 = a^2 - a^2 = 0$), we get the equation of this particular system of cones, viz.

$$\frac{x^2}{a_0^{\,2}} + \frac{y^2}{b_0^{\,2}} + \frac{z^2}{c_0^{\,2}} = 0,$$

the axes of ξ, η, and ζ coinciding here with those of x, y, and z, and $a_0^{\,2}$, $b_0^{\,2}$, $c_0^{\,2}$ being equal respectively to $a^2 + \delta_0$, $b^2 + \delta_0$, $c^2 + \delta_0$, δ_0 being the general value of δ.

This equation is *à priori* evident, for the annihilation of the absolute term in the common equation of a system of confocal surfaces gives that of a system of confocal cones, which are the asymptotic cones to the system of surfaces, and which have obviously for their common focal lines the asymptotes of the focal hyperbola.

As in every other system of confocal cones, these cones intersect two and two at right angles, which is manifest from their equation, or from the known property, that the tangent plane through any side of a cone of the second order makes equal angles with the two vector planes which contain each that side and one of the focal lines of the cone, from which it immediately appears that the two tangent planes through any one of the four intersecting sides of any two confocal cones (which lines evidently make equal angles with the common axes of these cones) make equal angles both with the same two planes, and are therefore at right angles to each other.

24. If the point O be at an infinite distance then will one of the three confocal surfaces which pass through it (the ellipsoid) be an infinite sphere, whose normal at that point will therefore pass through the centre, and the other two (the hyperboloids) will there coincide with their asymptotic cones, and will therefore have at that point the same normals as these cones. Hence, for a point at infinity, the principal axes may be found by a geometric construction quite elementary.

For, draw through the centre of gravity a line in the direction of the infinitely distant point, this will be one of the principal axes for that point; the others will be, of course, in a plane perpendicular to it: to find their directions in that plane, draw through the line two planes passing through the asymptotes to the focal hyperbola of the ellipsoid of gyration, and then two other planes bisecting the two supplemental angles, acute and obtuse, between the former; these latter will intersect the infinitely distant plane in the principal axes sought.

If the direction line of the infinitely distant point be one of the asymptotes themselves, the above construction failing leaves the remaining principal axes indeterminate: but that is precisely what ought to take place, for, at an infinitely distant point, the focal hyperbola coincides with its asymptote, and all points of that curve admit of an infinite number of principal axes.

In the case of an infinitely distant point, the analytic determination of the principal axes is also complete, which, as depending on the solution of a cubic equation, can, in the general case, be hardly considered as such.

For one of the roots of the cubic (VIII.) which gives the three values of δ, (on which everything depends), being in this case infinite, that equation is immediately depressible into a quadratic, and therefore completely solvible.

To find that quadratic, let $\alpha\,\beta\,\gamma$ be the direction angles of the infinitely distant point, and for $x\,y\,z$, in equation VIII., substituting $r\cos\alpha$, $r\cos\beta$, $r\cos\gamma$, let that equation be divided by r^2, and that quantity then made infinite, this will give

$$\frac{\cos^2\alpha}{a^2+\delta}+\frac{\cos^2\beta}{b^2+\delta}+\frac{\cos^2\gamma}{c^2+\delta}=0\ldots\ldots\ldots\text{(XII.)}$$

the quadratic for δ whose roots are the values required.

For an infinitely distant point, the system of enveloping cones (XI.), the reciprocals of the equimomental system of cones for that point will degenerate obviously into a system of cylinders; these being coaxal and confocal, as in the general case, will intersect any plane perpendicular to their common axis in a system of concentric coaxal and confocal conics. Hence, in a body, if we orthographically project upon any plane taken at random, the whole system of surfaces of the second order confocal with its ellipsoid of gyration, the outlines of the projections will form a system of confocal conics, of which the common centre will be the projection of the centre of gravity.

If from any point on the circle locus of the intersections of its pairs of rectangular tangents, we draw two tangents to one of these conics selected at will in the plane of projection, a plane drawn through the point perpendicular to one of these tangents will obviously touch the projected surface. Hence, by (VII.), A_0 being the primary semiaxis of that surface, and r_0 the distance of the point from the centre of gravity, which distance is obviously the same for all points of the circle in question, we have moment of inertia round that tangent $= Mr_0^2 - M(a_0^2 - a^2) =$ a constant quantity. Hence we know that all the tangents to any one of the projected conics are equimomental axes.

And conversely, the envelope in any plane taken at random in a body of a system of equimomental axes will be a conic, an ellipse, or hyperbola, as the case may be. And by giving different values to the common moment of inertia, the whole system of conics in the same plane will be confocal with the orthographic projection on that plane of the ellipsoid of gyration. This, it will be remembered, was proved before from other principles in (9), and as its consequences were there discussed at some length, we need here consider the property no further.

If the infinitely distant point be on the focal hyperbola, the enveloping cylinders being the limit to a system of cones of revolution, will be all of revolution round the focal asymptote, and the projection of the surfaces on any plane, perpendicular to that asymptote, will be all concentric circles, which last was also proved in the article just referred to.

Since for an infinitely distant point a_1 is infinite, the equation of the equimomental system of cones for such a point will be reduced to $\xi^2 = 0$ (as it ought from the analogy of the general case) except in the cases where a_0 is also nearly infinite; these cases are infinite in number and for them all, since the difference between two infinities does not necessarily vanish, we shall have for the curves of intersection of the cones by planes $\xi =$ a very large constant, the equation $\eta^2 + \zeta^2 =$ a constant, which curves, for different values of the constant, are therefore a system of concentric circles.

The directions of the principal axes depending all the while on $a_{\prime\prime}$ and $a_{\prime\prime\prime}$ which are finite and accurately determinable, we have, in the omission of these quantities in comparison with others infinitely great, the explanation of the apparent paradox noticed in (7).

25. In the general case, the point O being anywhere at all. If in the equations of the two systems of cones (X.) and

(XI.) we give to a_0 the particular value $a_0 = a_{\prime\prime}$, the equimomental cone (X.) will degenerate into two planes, the common cyclic planes, viz. of that system of cones and the corresponding enveloping cone, consequently into two right lines, these lines, there is no difficulty in seeing, are the two rectilinear generatrices of the hyperboloid of one sheet $a_{\prime\prime}$, confocal with the ellipsoid of gyration which passes through O.

In fact, once we know that the enveloping cones are confocal, we readily see what must be their focal lines; but the latter admits of a simple geometric solution, which at the same time proves directly the former.

For, if through any line, tangent to a surface of the second order, we draw two tangent planes to any confocal surface, these planes will make equal angles with the tangent plane to the original surface through the same touching line. This is manifest, for that tangent plane is a principal plane of the cone, which, from its point of contact as vertex, envelopes the confocal surface, and two tangent planes to a cone which intersect in a principal plane make equal angles with that plane.

Making now any point upon an hyperboloid of one sheet the vertex of a cone enveloping any confocal surface, let two planes through the generatrices of the hyperboloid at the point intersect in any side of the cone, these planes being tangent planes to the hyperboloid will make equal angles with the tangent plane to the confocal surface which is drawn through the side of contact, that is, with the tangent plane to the cone through that side.

The generatrices, therefore, possess with respect to the enveloping cone the property, that the vector planes passing through them and intersecting in any side of the cone, make equal angles with the tangent plane through that side. And this being the known property of the focal lines in a cone of the second order tells us that the cones, which from any vertex envelope a system of confocal surfaces, are confocal, having all the same focal lines, viz. the generatrices of the confocal hyperboloid of one sheet which passes through their vertex.

If in the general equation (XI.) for any point O, we put in successively for a_0^2 the quantities $a^2 - c^2$, $a^2 - b^2$, and $a^2 - a^2$, we shall get the three particular cones which from that vertex pass through the focal conics of the ellipsoid of gyration; these cones have been called, by Professor Mac Cullagh, *focal cones*, with respect to the surface of the point from which they diverge; the last of the three is

of course imaginary as passing through an imaginary curve, but the other two, as passing through the focal ellipse and hyperbola, are real: like every other pair of confocal cones they intersect at right angles; being coaxal, their four lines of intersection (which as passing through the two focal conics are called *bifocal lines*) make equal angles with each of the three common axes, that is, with each of the three principal axes at their vertex; and if through the centre three planes be drawn perpendicular to these same three axes, they also will make equal angles with the four bifocal lines, and will intercept on each of those lines three portions, which, measured from the vertex, are equal to the primary semiaxes of the three confocals which pass through that point, the four equal portions intercepted on the four lines by each plane being equal to the primary semiaxes of the surface to whose normal that plane is perpendicular.

These theorems (which readily appear from the equations of the two cones) are due to Professor Mac Cullagh, who has given them in his Examination papers, Dublin University Calendar: indeed the necessary limits of a paper like the present will oblige us, in what follows, to assume for the most part the known properties of confocal surfaces, referring the reader, to whom they may not be familiar, to the above papers, to a paper on Surfaces of the Second Order by the same author, in the Proceedings of the Royal Irish Academy, Part VIII., and to the memoir on the subject in Chasles' History of Geometry.

26. The following geometric construction, evident from the above principles, for determining the three principal axes at any given point O of a body, and also the three quantities $a_{,}$, $a_{,,}$, $a_{,,,}$, which give immediately (IX.) the three principal moments for the same, is perhaps the simplest that the nature of the case admits of, when we consider that the analytic solution of the same problem depends upon finding the roots of a cubic equation.

From O as vertex describe two cones passing through the focal ellipse and hyperbola of the ellipsoid of gyration, and intersecting therefore in four bifocal lines: the three pairs of planes which pass through these four lines will then intersect in the principal axes of O, and the three planes through the centre of gravity perpendicular to the three axes thus determined will intercept on each of the same four lines three portions which measured from O are equal to the primary semi-axes of the three confocals which pass through that point, that is, to $a_{,}$, $a_{,,}$, $a_{,,,}$, the three quantities required.

This construction possesses over those of (18) and (19) the advantages not only of simplicity and of giving the magnitudes of the principal moments at the same time with the directions of the principal axes, but also of being itself necessary to the completion of both; for to find the axes of a given cone of the second order, the simplest geometric construction is perhaps the following, which is immediately and evidently derivable from the above property of every system of two cones, which from a common vertex pass through two conics, of which one is the focal of the other: taking arbitrarily any plane section of the cone, describe the conic focal of that section; this will pierce the cone in four points; join these with the vertex, and through the four joining lines send the three pairs of planes which contain them two and two; each pair will intersect in an axis of the cone.

In the particular case when the point is at an infinite distance, we have seen (24) that the construction for principal axes becomes exceedingly simple, and other particular cases will be presently noticed in which the construction is also elementary: of this the reason is evident; for in the general case, to find the principal axes at a given point depends on finding the axes of a surface of the second order; which problem, though always reducible to finding the axes of a cone, cannot be solved by elementary geometry: on the contrary, in all the particular cases for which the construction is simple, we shall see that one principal axis is always given, and thus to find the others depends only on finding the axes of a given conic, and for this the construction is of course elementary.

[*To be continued*].

ON THE INTEGRATION OF CERTAIN EQUATIONS IN FINITE DIFFERENCES.

By the Rev. BRICE BRONWIN.

THE method of integrating certain equations in finite differences, which is illustrated in this paper by a few examples, is, I believe, quite new. The process is moreover very simple and easy. A few subsidiary formulæ are required, which must first be given.

$$\left.\begin{aligned} x\Delta^n y &= \Delta^n\{(x-n)y\} - n\Delta^{n-1}y \\ x^2\Delta^n y &= \Delta^n\{(x-n)^2y\} - n\Delta^{n-1}\{(2x-2n+1)y\} + n(n-1)\Delta^{n-2}y \end{aligned}\right\}\ldots(a).$$

Substitute for

$$\Delta^n\{(x-n)y\},\ \Delta^n\{(x-n)^2y\},\ \Delta^{n-1}\{(2x-2n+1)y\}$$

their values given by the known theorem

$$\Delta^n (PQ) = Q\Delta^n P + \frac{n}{1}\,\Delta Q\Delta^{n-1}P_1 + \frac{n(n-1)}{1.2}\,\Delta^2 Q\Delta^{n-2}P_2 + \text{\&c.},$$

and these formulæ will be verified,

$$\left.\begin{array}{r} x\Delta^{-n}y = \Delta^{-n}\{(x+n)\,y\} + n\Delta^{-n-1}y \\ x^2\Delta^{-n}y = \Delta^{-n}\{(x+n)^2 y\} + n\Delta^{-n-1}\{(2x+2n+1)\,y\} + n(n+1)\Delta^{-n-2}y \end{array}\right\}\ldots(b).$$

These may be verified in a similar manner to the last by the known theorem

$$\Sigma^n (PQ) = Q\Sigma^n P - \frac{n}{1}\,\Delta Q\Sigma^{n+1}P_1 + \frac{n(n+1)}{1.2}\,\Delta^2 Q\Sigma^{n+2}P_2 - \text{\&c.}$$

We are now prepared to proceed to some examples.

Let $(1-x^2)\,\Delta^2 y + p\,(p+1)\,y = 0$, ($p$ a positive integer)...(1).

Make $y = \Delta^{p-1}u$, and the above becomes

$$\Delta^{p+1}u - x^2\Delta^{p+1}u + p\,(p+1)\,\Delta^{p-1}u = 0.$$

Putting for $x^2\Delta^{p+1}u$, its value given by the second of (a), there results

$$\Delta^{p+1}\{1-(x-p-1)^2\}\,u + (p+1)\,\Delta^p\,(2x-2p-1)\,u = 1\,;$$

$$\text{or } \Delta\,\{1-(x-p-1)^2\}\,u + (p+1)(2x-2p-1)\,u = \Delta^{-p}0.$$

But we will neglect $\Delta^{-p}0$, and find each particular integral separately. The last result may be put under the form

$$\{1-(x-p)^2\}\,u_{x+1} - \{1+p+p^2-x^2\}\,u_x = 0.$$

Make $\dfrac{1+p+p^2-x^2}{1-(x-p)^2} = f(x+1)$. Then changing x into $x-1$, we have $u_x - f(x)\,u_{x-1} = 0$; and integrating we obtain

$$u_x = Cf(x)f(x-1)f(x-2)\ldots. = CPf(x) \text{ suppose,}$$

which is the integral always given.

Again, make $y = \Delta^{-p-2}u$. With this value (1) becomes

$$\Delta^{-p}u - x^2\Delta^{-p}u + p\,(p+1)\,\Delta^{-p-2}u = 0.$$

Put for $x^2\Delta^{-p}u$ its value from the second of (b), and we have

$$\Delta^{-p}\{1-(x+p)^2\}\,u - p\Delta^{-p-1}(2x+2p+1)\,u = 0,$$

$$\text{or}\quad \Delta\,\{1-(x+p)^2\}\,u - p\,(2x+2p+1)\,u = 0\,;$$

which, as before, may be put under the form

$$\{1-(x+p+1)^2\}\,u_{x+1} - (1+p+p^2-x^2)\,u_x = 0.$$

Make $\dfrac{1+p+p^2-x^2}{1-(x+p+1)^2} = f_1(x+1)$, and change x into $x-1$; the last equation becomes $u_x - f_1(x)\,u_{x-1} = 0$, which gives $u_x = C_1Pf_1(x)$.

Therefore the complete integral of (1) is

$$y = C\Delta^{p-1}Pf(x) + C_1\Delta^{-p-2}Pf_1(x).$$

Let $x^2\Delta^2 y + m\Delta y - p(p+1)y = 0$, ($p$ a positive integer)...(2).

Here again make $y = \Delta^{p-1}u$, and we have

$$x^2\Delta^{p+1}u + m\Delta^p u - p(p+1)\Delta^{p-1}u = 0;$$

Or, by putting for $x^2\Delta^{p+1}u$ its value given by (a),

$$\Delta^{p+1}(x-p-1)^2 u - (p+1)\Delta^p(2x-2p-1)u + m\Delta^p u = 0;$$

or $\Delta(x-p-1)^2 u - \{(p+1)2x - (p+1)(2p+1) - m\}u = 0$;

and $\quad (x-p)^2 u_{x+1} - \{x^2 - p(p+1) - m\}u_x = 0.$

Make $\dfrac{x^2 - p(p+1) - m}{(x-p)^2} = f(x+1)$; and we find, as before,

$$u_x - f(x)u_{x-1} = 0,\ u_x = CPf(x).$$

Again make $y = \Delta^{-p-2}u$, and (2) will become

$$x^2\Delta^{-p}u + m\Delta^{-p-1}u - p(p+1)\Delta^{-p-2}u = 0.$$

By the same steps as before, we have

$$\Delta^{-p}(x+p)^2 u + p\Delta^{-p-1}(2x+2p+1)u + m\Delta^{-p-1}u = 0,$$

$$\Delta(x+p)^2 u + (2px + 2p^2 + p + m)u = 0,$$

$$(x+p+1)^2 u_{x+1} - \{x^2 - p(p+1) - m\}u_x = 0.$$

And if $\dfrac{x^2 - p(p+1) - m}{(x+p+1)^2} = f_1(x+1)$; then $u_x - f_1(x)u_{x-1} = 0$,

and $u_x = C_1 Pf_1(x)$. Therefore the complete integral of (2) is

$$y = C\Delta^{p-1}Pf(x) + C_1\Delta^{-p-2}Pf_1(x).$$

Let $\quad \Delta^2 y + mx\Delta y + pmy = 0$ . . . . (3).

Make $y = \Delta^{p-1}u$; then we have successively

$$\Delta^{p+1}u + mx\Delta^p u + pm\Delta^{p-1}u = 0,$$

$$\Delta^{p+1}u + m\Delta^p(x-p)u = 0,$$

$$\Delta u + m(x-p)u = \Delta^{-p}0;$$

which we know how to integrate, and which will give the complete integral of the proposed at once. We cannot in this example obtain two particular integrals separately. The supernumerary arbitraries contained in

$$\Delta^{-p}0 = \Delta^{-p+1}a = \Delta^{-p+2}(ax+b) = \&c.,$$

must be determined by substituting the value found for y in the given equation. And the same thing must be done with regard to $C_1\Delta^{-p-2}Pf(x)$ in examples (1) and (2), and in all similar cases.

Let $\quad x\Delta^2 y + mx\Delta y + pmy = 0$ . . . . (4).

Make $y = \Delta^{p-1}u$; then $x\Delta^{p+1}u + mx\Delta^p u + pm\Delta^{p-1}u = 0$,

and $\Delta^{p+1}(x-p-1)u-(p+1)\Delta^p u+m\Delta^p(x-p)u=0$,

$$\Delta(x-p-1)u-(1+p+mp-mx)u=\Delta^{-p}0,$$

or $$(x-p)\Delta u-(p+pm-mx)u=\Delta^{-p}0.$$

As we know how to integrate this, I shall not proceed with it further.

Let $(1+x^2)\Delta^2 y+mx\Delta y+p(m-p-1)y=0\ldots.(5)$, where it must be remembered p, as always, is a positive integer. Make $y=\Delta^{p-1}u$, and we have by the same steps as heretofore

$$\Delta^{p+1}u+x^2\Delta^{p+1}u+mx\Delta^p u+p(m-p-1)\Delta^{p-1}u=0,$$

$$\Delta^{p+1}u+\Delta^{p+1}(x-p-1)^2u-(p+1)\Delta^p(2x-2p-1)u+m\Delta^p(x-p)u=0,$$

$$\Delta\{(x-p-1)^2+1\}u-\{(2p-m+2)x-(p+1)(2p+1)+pm\}u=\Delta^{-p}0,$$

or $\{(x-p)^2+1\}u_{x+1}-\{x^2-mx-p(p-m+1)+1\}u_x=\Delta^{-p}0$, which will give the complete integral. We cannot find two particular ones separately.

The next three examples give only particular integrals.

Let $$\Delta^2 y-mx\Delta y+pmy=0\ldots\ldots\ldots\ldots(6).$$

Make $y=\Delta^{-p-1}u$; then $\Delta^{-p+1}u-mx\Delta^{-p}u+pm\Delta^{-p-1}u=0$,

and $$\Delta^{-p+1}u-m\Delta^{-p}(x+p)u=0,$$

whence we obtain either

$$\Delta u-m(x+p)u=0,\quad \text{or}\quad \Delta u-m(x+p)u=\Delta^p 0,$$

$\Delta^p 0$ is an actual number; but as the characteristic Δ has reference only to the variable x, we must here make $\Delta^p 0=0$, and therefore shall only have a particular integral.

Let $$x\Delta^2 y-mx\Delta y+pmy=0\ldots\ldots\ldots(7).$$

Make $y=\Delta^{-p-1}u$, and the proposed becomes

$$x\Delta^{-p+1}u-mx\Delta^{-p}u+pm\Delta^{-p-1}u=0,$$

Or $$\Delta^{-p+1}(x+p-1)u-\Delta^{-p}(mx+pm-p+1)u=0,$$

and $$\Delta(x+p-1)u-(mx+pm-p+1)u=0.$$

Or $$(x+p)u_{x+1}-(mx+x+pm)u_x=0,$$

which gives a particular integral.

Let $(1+x^2)\Delta^2 y+mx\Delta y-p(m+p-1)y=0\ldots.(8)$.

Make $y=\Delta^{-p-1}u$; then

$$\Delta^{-p+1}u+x^2\Delta^{-p+1}u+mx\Delta^{-p}u-p(m+p-1)\Delta^{-p-1}u=0,$$

and $\Delta^{-p+1}u+\Delta^{-p+1}(x+p-1)^2u$

$$+\Delta^{-p}\{(2p+m-2)x+(p-1)(2p-1)+pm\}u=0,$$

$$\Delta\{(x+p-1)^2+1\}u+\{(2p+m-2)x+(p-1)(2p-1)+pm\}u=0,$$

$$\{(x+p)^2+1\}u_{x+1}-\{x^2-mx-p(p+m-1)-1\}u_x=0.$$

We know how to find the other particular integral by the aid of the one obtained, but could not in these examples find it independently without transforming the equations, and perhaps not by that means.

In order to give an example or two in which x^3 enters into the coefficients; we need, in addition to the formulæ (a) and (b), the following,

$$x^3\Delta^n y = \Delta^n\{(x-n)^3 y\} - n\Delta^{n-1}\{(x-n)(3x-3n+1)y\} + n(n-1)\Delta^{n-2}\{(3x-3n+3)y\} - n(n-1)(n-2)\Delta^{n-3}y \ldots (c).$$

This and the following may be verified in exactly the same way as (a) and (b).

$$x^3\Delta^{-n}y = \Delta^{-n}\{(x+n)^3 y\} + n\Delta^{-n-1}\{(x+n)(3x+3n+1)y\} + n(n+1)\Delta^{-n-2}\{(3x+3n+3)y\} + n(n+1)(n+2)\Delta^{-n-3}y \,.\,.(d).$$

We now proceed to some examples in which these formulæ are required.

Let $$x^3\Delta^2 y + 2px^2\Delta y + p(p-1)(1+x)y = 0 \ldots .(9).$$

Make $y = \Delta^{p-2}u$; and the above becomes

$$x^3\Delta^p u + 2px^2\Delta^{p-1}u + p(p-1)\Delta^{p-2}u + p(p-1)\,x\Delta^{p-2}u = 0.$$

Substituting for $x^3\Delta^p u$ its value given by (c); and for $x^2\Delta^{p-1}u$, $x\Delta^{p-2}u$ their values from (a), there results

$$\Delta^p(x-p)^3 u - p\Delta^{p-1}(x^2-2px-3x+p^2+3p-2)u = 0.$$

Or $$\Delta(x-p)^3 u - p(x^2-2px-3x+p^2+3p-2)u = \Delta^{-p+1}0,$$

which will give the complete integral.

Let $$x^3\Delta^2 y - 2px^2\Delta y + p(p+1)(1+x)y = 0 \ldots .(10).$$

Make $y = \Delta^{-p-2}u$; then

$$x^3\Delta^{-p}u - 2px^2\Delta^{-p-1}u + p(p+1)\Delta^{-p-2}u + p(p+1)x\Delta^{-p-2}u = 0;$$

whence $$\Delta^{-p}(x+p)^3 u + p\Delta^{-p-1}(x^2+2px-3x+p^2-3p-2)u = 0,$$

and $$\Delta(x+p)^3 u + p(x^2+2px-3x+p^2-3p-2)u = 0.$$

This will only give a particular integral. It is worthy of remark, that in the two last examples, and in every step of their solutions, p has contrary signs. It is the same in the two following, which shall conclude this paper.

Let $$x^3\Delta^3 y + p(p-1)(p-2)y = 0 \ldots\ldots (11).$$

Make $y = \Delta^{p-3}u$, and we have $x^3\Delta^p u + p(p-1)(p-2)\Delta^{p-3}u = 0$,

Or $$\Delta^p(x-p)^3 u - p\Delta^{p-1}(x-p)(3x-3p+1)u + p(p-1)\Delta^{p-2}(3x-3p+3)u = 0,$$

and $$\Delta^2(x-p)^3 u - p\Delta(x-p)(3x-3p+1)u + 3p(p-1)(x-p+1)u = \Delta^{-p+2}0.$$

The solution of this will give the complete integral of the given equation, which is therefore reduced to the integration of one of the second order.

Let $$x^3\Delta^3 y - p(p+1)(p+2)y = 0 \ldots\ldots (12).$$

Make $y = \Delta^{-p-3}u$; then $x^3\Delta^{-p}u - p(p+1)(p+2)\,\Delta^{-p-3}u = 0$,

$$\Delta^{-p}(x+p)^3 u + p\Delta^{-p-1}(x+p)(3x+3p+1)\,u$$
$$+ p(p+1)\,\Delta^{-p-2}(3x+3p+3)\,u = 0,$$

and $$\Delta^2(x+p)^3 u + p\Delta(x+p)(3x+3p+1)\,u$$
$$+ 3p(p+1)(x+p+1)\,u = 0.$$

The complete solution of this will only give two particular integrals of the given equation.

We might employ more complex forms of transformation than those used in this paper; and a method exactly similar may be applied to the solution of similar differential equations. It will be easy to infer that method from what is done here, but I have sent a few examples in explanation of it to the *Mathematician*, which perhaps may soon appear in that publication.

ON SYMBOLICAL GEOMETRY.

By Sir William Hamilton.

(*Continued.*)

Other Interpretation of the Associative Principle of Multiplication: Theorem of the two Conjugate Transversals of a Spherical Quadrilateral (which are the Cyclic Arcs of a circumscribed Spherical Conic).

18. The theorem of the two hexagons gives also the following theorem: If upon each of the four sides of a spherical quadrilateral, or on that side prolonged, a portion be taken commedial with the side (two arcs being said to be *commedial* when they have one common point of bisection); and if four extreme points of the four portions thus obtained be ranged on one transversal arc of a great circle, in such a manner that the part of this arc comprised between the first and third sides is commedial with the part comprised between the second and fourth: then the four other extremities of the same four portions will be ranged on another great circle; and the parts of this second or *conjugate* transversal, which are intercepted respectively by the same two pairs of opposite sides of the quadrilateral, will be in like manner commedial with each other.

For let the corners of the quadrilateral be denoted by the letters A, B, C, D, and let the side from A to B be cut in two points A' and B'', while the three other sides are cut in three other pairs of points, which may be called B' and C'', C' and D'', and D' and A'' respectively. Then, if the arcs from A' to C' and from B' to D' be commedial portions of one common great circle, or of a first transversal arc, the arcs from A' to B' and from D' to C' will be *symbolically equal arcs*, in the sense of the preceding article; and therefore, in the notation of that article, we may now write the equation

$$\frown B'A' = \frown C'D' \text{ .... } (133).$$

In like manner the conditions, that the four portions of the sides of the quadrilateral shall be commedial with the sides themselves, give the four other equations of the same kind,

$$\left.\begin{array}{ll} \frown A'A = \frown BB''; & \frown B'B = \frown CC''; \\ \frown C'C = \frown DD''; & \frown D'D = \frown AA''. \end{array}\right\} \text{ .. } (134).$$

Hence, by alteration and inversion, we find that the five successive sides

$$\frown AB'', \ \frown D'A, \ \frown C'D', \ \frown CC', \ \frown C''C,$$

of the spherical hexagon $B''AD'C'CC''$ are respectively and symbolically equal to the five successive diagonals

$$\frown A'B, \ \frown DA'', \ \frown B'A', \ \frown D''D, \ \frown BB',$$

of the other hexagon $BA''A'DB'D''$; and therefore, by the theorem of the two hexagons, the sixth side of the former figure must be symbolically equal to the sixth diagonal of the latter; that is, we may write the symbolical equation,

$$\frown B''C' = \frown A''D'' \text{ .... } (135).$$

But this expresses a relation equivalent to the following, that the two arcs from A'' to C'' and from B'' to D'' are commedial portions of one common great circle, or second transversal arc, which was the thing to be proved.

Reciprocally, the associative principle of geometrical multiplication, in so far as it relates to the directions of straight lines in space, may be expressed by the assertion that the symbolical equation between arcs (135) is a consequence of the five other equations of the same kind (133) and (134); this principle of symbolical geometry may therefore be so interpreted as to coincide with the foregoing *theorem of the two conjugate transversals* of a spherical quadrilateral, instead of the theorem of the two spherical hexagons. It is easy to see that to a given quadrilateral correspond infinitely many such pairs of conjugate transversal arcs; and those readers

who are familiar with the theory of *spherical conics** will recognise in these conjugate transversals, A′B′C′D′, A″B″C″D″, the two *cyclic arcs* of such a conic, circumscribed about the proposed quadrilateral ABCD; but it suits better the plan of this communication on symbolical geometry to pass at present to another view of the subject.

It may however be noticed here, that in the first of the two hexagons already mentioned, *any two pairs of opposite sides intercept commedial portions on either of the two sides remaining;* and that the associative principle asserts that *if* a spherical hexagon have *five* of its sides thus *cut commedially*, the *sixth* side also will be cut in the same way. Or, because the two sets of alternate diagonals of the second hexagon are sides of two triangles, which have for their corners the alternate corners of this hexagon, we may in another way eliminate this second hexagon, and may express the same principle of spherical geometry by saying, that *if one set of alternate sides of a* (first) *spherical hexagon*, taken in their order (as first, third, and fifth), *be respectively and symbolically equal to the three successive sides of a triangle*, then the *other set of alternate sides of the same hexagon will be* in like manner *symbolically equal to the sides of another triangle.* This last interpretation of the associative principle is even more immediately suggested than any other, by the forms of the

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* The plane of the first side of the quadrilateral, or the plane of OAB, if O denote the centre of the sphere, is cut by the plane of the first transversal arc in the radius A′O, and by the plane of the second transversal arc in the radius B″O. Thus the four plane faces of the tetrahedral angle, of which the four edges are the four radii from O to the four corners A, B, C, D of the quadrilateral, are cut by any secant plane parallel to the plane of the first transversal arc in four indefinite straight lines, which are respectively parallel to the four other radii A′O, B′O, C′O, D′O of the sphere; and consequently, in virtue of the equation (133), between the arcs which these last radii include, these four new lines in one common secant plane have the angular relation required for their being the (prolonged) sides of a (plane) quadrilateral inscribed in a circle; therefore the four edges of the same tetrahedral angle are cut by the same secant plane in points which are on the circumference of a circle; therefore they are edges or sides of a cone which has this circle for its base, and has its vertex at the centre of the sphere. But the intersection of such a cone with such a concentric sphere is called a *spherical conic;* a plane through its vertex, parallel to its circular base, is called a *cyclic plane;* and the intersection of this latter plane with the sphere has received the designation of a *cyclic arc*. Therefore the first transversal arc A′B′C′D′ is (as asserted in the text) a cyclic arc of a spherical conic circumscribed about the quadrilateral ABCD: and by a reasoning of exactly the same kind it may be proved, that the second transversal A″B″C″D″ is another cyclic arc of the same conic, or that its plane is a second cyclic plane, being parallel to the plane of another (or *subcontrary*) circular section.

equations (131) (132); in the notation of the present article, *the two triangles* are BA'B' and A''DD'', which may be considered as having their *bases* A'B' and A''D'' *on the two cyclic arcs* above alluded to, while their *vertical angles* at B and D may be said to be *angles in the same segment* (or in alternate segments) *of the spherical conic:* since, by (134), the two arcual sides BA', BB' of the one angle intersect respectively the two sides DA'', DD'' of the other angle, in the points A and C, which points of intersection, as well as the vertices B and D, are corners of the quadrilateral inscribed in that spherical conic.

Symbolical Addition of Arcs upon a Sphere; Associative and Non-commutative Properties of such Addition.

19. The foregoing geometrical interpretations of the associative principle or property of the multiplication of geometrical fractions, may assist us in forming and applying the conception of the symbolical addition of arcs of great circles upon a sphere, and in establishing and interpreting an analogous principle or property of such symbolical addition.

As it has been already proposed in the third article of this paper, and also in the works of other writers on subjects connected with the present, to adopt, for the *addition of straight lines having direction,* a rule expressed by the formula

$$\mathrm{CB} + \mathrm{BA} = \mathrm{CA} \quad \ldots\ldots\ldots\ldots (7),$$

in whatever manner the three points ABC may be situated or related to each other; so it seems natural to adopt now, for the analogous *addition of arcs upon a sphere,* when directions as well as lengths are attended to, the corresponding formula,

$$\frown\mathrm{CB} + \frown\mathrm{BA} = \frown\mathrm{CA} \ldots\ldots (136).$$

Admitting this latter formula as *the definition of the effect of the sign + when inserted between two such symbols of arcs,* and granting also that it is permitted, in any such formula, to substitute for any arcual symbol another which is *equal* thereto, we shall have, by the two first and two last equations (134) respectively, the two following other equations,

$$\left.\begin{array}{l} \frown\mathrm{B''C''} = \frown\mathrm{AA'} + \frown\mathrm{B'C} \\ \frown\mathrm{A''D''} = \frown\mathrm{AD'} + \frown\mathrm{C'C} \end{array}\right\} \ldots\ldots (137).$$

The two sums in these second members will therefore be symbolically equal if we have the equation

$$\frown\mathrm{A'D'} = \frown\mathrm{B'C'} \quad \ldots\ldots\ldots (138),$$

because (135) has been seen to follow from (133) and (134). But by (136) and (138), we have the expression

$$\frown B'C = \frown A'D' + \frown C'C \quad \ldots\ldots (139);$$

consequently the associative principle of multiplication, considered in several recent articles, when combined with the *formula of arcual addition* (136), conducts to the following formula,

$$\frown AA' + (\frown A'D' + \frown C'C) = (\frown AA' + \frown A'D') + \frown C'C \ldots (140),$$

or, as it may be more concisely written,

$$\frown''' + (\frown'' + \frown') = (\frown''' + \frown'') + \frown' \ldots\ldots (141):$$

which in its form agrees with ordinary algebra, and may be said to express the *associative principle of the symbolical addition of arcs;* since the three arcs added in (140) or (141) may be any three arcs of great circles upon one common spheric surface. It is remarkable that so much geometrical meaning should be contained in so simple and elementary a form; for this form (141), which is *apparently an algebraic truism,* and has been here deduced from the associative principle of multiplication of geometrical fractions, may reciprocally be substituted for it, and therefore includes in its interpretation, *if we adopt the symbolical definition* (136) of the effect of + between two symbols of arcs, all those theorems respecting spherical great circles, triangles, quadrilaterals, hexagons, and conics, which have been deduced or mentioned as geometrical results of the associative principle in the two foregoing articles. And this encouragement to adopt the foregoing very simple definition (136) of the meaning of a symbol such as $\frown'' + \frown'$, is the more worthy of attention, because the *same definition* conducts to a *departure from the ordinary rules of symbolical addition* in *another* important point; since, when combined with the *definition of symbolical equality between arcs* assigned in the 17th article, it shews that *addition of arcs is in general a non-commutative operation.* For if we conceive two arcs of different great circles on one sphere, from A to B and from C to D, to bisect each other in a point E, we shall then have the two symbolical equations

$$\frown AE = \frown EB, \quad \frown CE = \frown ED \quad \ldots\ldots (142);$$

and therefore, whereas by (136),

$$\frown AE + \frown ED = \frown AD \quad \ldots\ldots (143),$$

the result of the addition of the same two arcs, taken in a different order, will be

$$\frown ED + \frown AE = \frown CB \quad \ldots\ldots (144).$$

And although the two *sum-arcs*, $\frown$AD and $\frown$CB, thus obtained, connecting two opposite pairs of extremities of the two commedial arcs $\frown$AB and $\frown$CD, are *equally long*, yet they are in general *parts of different great circles*, and therefore *not symbolically equal* in the sense of the 17th article. This result, which may at first sight seem a paradox, illustrates and is intimately connected with the analogous result obtained in the 13th article, respecting the general non-commutativeness of geometrical multiplication; for we shall find that there exists a species of *logarithmic connexion* between arcs situated in different great circles on a sphere and fractional factors belonging to different planes, which is analogous to, and includes as a limiting case, the known connexion between ordinary imaginary logarithms and angles in a single plane. It may be here remarked, that with the same definition (136) *in any symbolical addition of three successive arcs, the two partial sum-arcs,*

$$\frown'' + \frown' \text{ and } \frown''' + \frown'' \;\ldots\ldots\; (145),$$

are portions of the cyclic arcs of a certain spherical conic, circumscribed about a quadrilateral which has

$$\frown', \frown'', \frown''', \text{ and } \frown''' + \frown'' + \frown' \;\ldots\; (146),$$

that is, *the three proposed summand-arcs and their total sum-arc, for portions of its four sides*, or of those sides prolonged; as will appear by supposing that the three summands, $\frown'$, $\frown''$, $\frown'''$, coincide respectively with the arcs $\frown$CC′, $\frown$B′C, $\frown$AA′, in the notation of the preceding article.

[*To be continued.*]

ON THE THEORY OF INVOLUTION IN GEOMETRY.

By Arthur Cayley.

When three conics have the same points of intersection, any transversal intersects the system in six points, which are said to be in involution. It appears natural to apply the term to the conics themselves; and then it is easy to generalize the notion of involution so as to apply it to functions of any number of variables. Thus, if $U, V\ldots$ be homogeneous functions of the same order of any number of variables $x, y\ldots$. A function Θ, which is a linear function of $U, V\ldots$, is said to be in involution with these functions. More generally Θ may be said to be in involution with any

system of factors of these functions: or if $U, V\ldots$ be given functions of $x, y, z\ldots$, homogeneous of the degrees $m, n\ldots$, and $u, v\ldots$ arbitrary homogeneous functions of the degrees $r-m$, $r-n\ldots$; then, if $\Theta = uU + vV + \ldots$, Θ is a function of the degree r, which is in involution with $U, V\ldots$. The question which immediately arises, is to find the degree of generality of Θ, or the number of arbitrary constants which it contains. And this is a question which, from the variety and interest of its geometrical interpretations, has very frequently been treated of by geometers, though never, I believe, in quite so general a form, (the number r has almost always had particular values given to it, except in a short paper of my own, on the particular case of two curves, in the *Journal*, III. 211).* There is also an analytical application of the theory, of considerable interest, to the problem of elimination between any number of equations containing the same number of variables. Suppose, for instance, two equations, $U = 0$, $V = 0$, when U, V are homogeneous functions of x, y of the degrees m, n respectively. To eliminate the variables it is sufficient to multiply the first equation by $x^{n-1}, x^{n-2}y\ldots, y^{n-1}$, and the second by $x^{m-1}\ldots, y^{m-1}$, and from the equations so obtained to eliminate linearly the $(m+n)$ quantities $x^{m+n-1}, x^{m+n-2}y\ldots, y^{m+n-1}$. But in the case of a greater number of equations it is not at first obvious how many new equations should be obtained; and when a number apparently sufficiently great have been found, it may happen that the equations so obtained are not independent, and that

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* The first suggestion of the problem is contained in a memoir of Euler's —"Sur une contradiction apparente dans la doctrine des lignes courbes." *Mem. de Berlin*, tom. IV. p. 219, 1748. It is noticed also in Cramer's *Introduction a l'analyse des lignes courbes*. The following memoirs also have been published on the subject. Plücker, "Recherches sur les courbes algebraiques de tous les degrès," *Gerg. Ann.* tom. XIX. p. 97; "Recherches sur les surfaces algebraiques de tous les degrès," p. 129. (A great number of memoirs on particular applications of the theory are contained in Gergonne.) "Jacobi de relationibus quæ locum habere debent inter puncta intersectionis duarum curvarum vel trium superficierum dati ordinis, simul cum enodatione paradoxi Algebraici."—*Crelle*, tom. XV. Plücker, "Theorèmes generaux concernant les equations d'un degrè quelconque entre un nombre quelconque d' inconnues."—*Crelle*, tom. XVI. (But this last must be read with caution, as several of the theorems are incorrect, or at least stated without the proper limitations.) And the *Einleitende Betrachtungen*, in Plücker's "Theorie der Algebraischen Curven." The following memoirs of Hesse, containing developments relative to the case of three surfaces of the second order, may likewise be mentioned, "De curvis et superficiebus secundi gradus," *Crelle*, tom. XX. p. 285; and "Ueber die lineare Construction des achten Schnitt-punctes dreier Oberflächen zweiter Ordnung, wenn Sieben Schnitt-puncte derselben gegeben sind," *Crelle*, tom. XXVI. p. 147.

the elimination cannot be performed. But in shewing the connexion that exists between these different equations, the theory of involution explains in what manner a system is to be formed, which includes all the really independent equations, and gives the means of detecting the extraneous factors which appear in the result of the linear elimination of the different terms of these; but I do not see at present any mode of obtaining the final result at once in its reduced form free from any extraneous factors.

Let X, Y, ... be given homogeneous functions of the same degree of any number of variables, and suppose

$$\Theta = aX + \beta Y + \dots,$$

a, β ... being constants, and the number of terms in the series being g; Θ contains therefore g arbitrary constants. If however, by giving to a, β... particular values a_1, β_1..., or a_2, β_2..., and representing by Θ_1, Θ_2... the corresponding values of Θ, we have *identically*

$$\Theta_1 = 0, \quad \Theta_2 = 0, \dots \text{ (}h\text{ equations)};$$

then the constants in Θ group themselves together into a smaller number $g - h$ of arbitrary constants. This supposes, however, that the equations (2) are linearly independent, if there are a certain number (k) of equations

$$\Phi_1 = 0, \quad \Phi_2 = 0 \dots,$$

(where Φ_1, Φ_2, ...are linear functions of Θ_1, Θ_2....) which are identically satisfied, independently of the equations (2), then the equations (2) are equivalent to $h - k$ equations, and the function Θ contains $g - (h - k)$ or $g - h + k$, arbitrary constants. Similarly if the functions Φ are not independent; so that the number of arbitrary constants really contained in Θ is always

$$N = g - h + k - \&c. \dots\dots$$

Consider now the case of a function Θ, homogeneous of the rth degree in the variables x, y.... {($\theta + 1$) in number}. Let U, V....be functions of the degrees m, n...., and suppose

$$\Theta = uU + vV + \dots.$$

where u, v.... are arbitrary functions of the degrees $r - m$, $r - n$,.....[r is supposed throughout greater than m, n....]. Suppose for shortness that the number of terms in the complete function of θ variables, and of the order ρ, or the quotient $\dfrac{[\rho + \theta]^\theta}{[\theta]^\theta}$ is represented by $[\rho, \theta]$.

Then the function Θ contains apparently a number

$$([r - m,\theta] + [r - n,\theta] + \ldots)$$

of arbitrary constants.

But since we should have identically $\Theta = 0$ by assuming $u = LV, v = -LU, w = 0$, &c... (L the general function of the order $r - m - n$), or $u = MW, v = 0, w = -MU$ (M the general function of the order $r - m - p$) &c., the number N must be diminished by

$$[r - m - n, \theta] + [r - m - p, \theta] + [r - n - p, \theta] + \ldots .$$

But the equations just obtained are themselves not linearly independent, and in consequence of this the number of arbitrary constants has to be increased by $[r - m - n - p, \theta] + \ldots$ and so on. So that finally the whole number of arbitrary constants in the function θ is

$$\begin{aligned} N = [r - m, \theta] + [r - n, \theta] + [r - p, \theta] + \ldots \\ - [r - m - n, \theta] - [r - m - p, \theta] - [r - n - p, \theta] - \ldots \\ + [r - m - n - p, \theta] + \ldots\ldots\ldots\ldots (A). \end{aligned}$$

This however supposes that all the numbers $r - m, r - n \ldots$ $r - m - n, \ldots$ are positive: whenever this is not the case for any one of them, the corresponding term is obviously to be omitted. With this convention the equation (A) gives always the correct number of arbitrary constants in Θ: it will be convenient to represent it in the abbreviated form

$$N = \{r : m, n, p, \ldots : \theta\}.$$

An expression analogous to this, for the particular case of $r = m$, but incorrect on account of the omission of all the terms after the second line, has been given by M. Plücker (*Crelle*, tom. p.), and even some of his particular formulæ are incorrect. But proceeding to examine some particular cases: if $r > m + n + p + \ldots - \theta - 1$, then in the expression (A) either no terms are to be omitted, or else the terms to be omitted reduce themselves to zero, so that N is given by this formula continued to its last term. It will be subsequently shewn that in this case

$$\{r : m, n, p \ldots : \theta\} = [r, \theta] - mnp \ldots$$

Or in the case of two or three variables, we have the theorem, "If a curve or surface of the order r be determined to pass through the mn points of intersection of two curves of the orders m and n, or the mnp points of intersection of three surfaces of the orders m, n, p; then if $r > m + n - 3$, or $r > m + n + p - 4$, the curve or surface contains precisely the same number of arbitrary constants as if the mn or mnp points were perfectly arbitrary."

This is natural enough; the peculiarity is in the case where $r \not> m + n - 3$, or $r \not> m + n + p - 4$. For instance, for two curves, $r \not> m + n - 3$, we have

$$\{r : m, n : 2\} = [r - m, 2] + [r - n, 2]$$
$$= [r, 2] - mn + [r - m - n, 2].$$

Or the new curve contains $\frac{1}{2}[m + n - r - 1]^2$ more arbitrary constants than it would do if the mn points, through which it was made to pass, had been perfectly arbitrary; a result given before in the *Journal*.

In the case of surfaces, if $r \not> m + n + p - 4$. Then assuming $r > m + n - 4$, $m + p - 4$, or $n + p - 4$, we have

$$\{r : m, n, p : 3\} = [r - m, 3] + [r - n, 3] + [r - p, 3]$$
$$- [r - m - n, 3] - [r - m - p, 3] - [r - n - p, 3]$$
$$= [r, 3] - mnp - [r - m - n - p, 3].$$

Or the surface contains $\frac{1}{6}[m + n + p - r - 1]^3$ more arbitrary constants than it would do if the mnp points, through which it was made to pass, had been perfectly arbitrary. Similarly, in the case where r is not greater than one or more of the quantities $m + n - 4$, $m + p - 4$, $n + p - 4$. Thus in particular, if r be not greater than the least of these quantities

$$\{r : m, n, p : 3\} = [r, 3] - mnp + [r - n - p, 3] + [r - m - p, 3]$$
$$+ [r - m - n, 3] - [r - m - n - p, 3].$$

Or the surface contains

$$\frac{1}{6}[m + n + p - r - 1]^3 + \frac{1}{6}[n + p - r - 1]^3 + \frac{1}{6}[m + p - r - 1]^3$$
$$+ \frac{1}{6}[m + n - r - 1]^3$$

more arbitrary constants than it would otherwise have done. Again, for a surface of the r^{th} order, subjected to pass through the curve of intersection of two surfaces of the orders m, n,

$$\{r : m, n, 3\} = [r - m, 3] + [r - n, 3] - [r - m - n, 3].$$

In which the last term, or $\frac{1}{6}[m + n - r - 1]^3$, is to be omitted when $r \not> m + n - 4$.

The function of the r^{th} order, which is satisfied by the systems of values which satisfy the equations of the orders $m, n. .$ contains, we have seen, $[r, m, n, p. . \theta]$ arbitrary constants; hence it may be determined so as to pass through this number, diminished by unity, of arbitrary points. But the equation being determined in general by the condition of being satisfied by $[r, \theta] - 1$ systems of variables, it will be completely determined if, in addition to the above number

of arbitrary systems, we suppose it to be satisfied by a number $N = [r, \theta] - \{r; m, n, p\,..:\theta\}$ of systems satisfying the equations above. Hence the theorem

"The equation of the r^{th} order which is satisfied by a number $N = [r, \theta] - \{r; m, n, p\,..:\theta\}$ of systems satisfying the equations of the orders $m, n, p\,..$ is satisfied by any systems whatever which satisfy these equations."

In particular—"The surface of the r^{th} order which passes through $[r, \theta] - \{r : m, n : \theta\}$ of points in the curve of intersection of two surfaces of the orders m, n—or through $[r, \theta] - [r; m, n, p : \theta\}$ of the mnp points of intersection of three surfaces of the orders m, n, p—passes through the curve of intersection, or through the mnp points of intersection."

Thus a surface of the second order which passes through eight points of the curve of intersection of two surfaces of the second order passes through this curve; and any surface of the second order which passes through seven of the points of intersection of three surfaces of the second order passes through the eighth point. (The first theorem obviously fails if the eight points have the relation in question, *i.e.* if they are the eight points of intersection of three surfaces of the second order.)

Again—"The curve of the r^{th} order which passes through $[r, \theta] - \{r : m, n : \theta\}$ of the points of intersection of two curves of the orders mn, passes through the remaining points of intersection." *e.g.* Any curve of the third order which passes through eight of the points of intersection of two curves of the third order, passes also through the ninth point.

Consider next the following question, which has been treated of by Jacobi in the memoir already quoted. "To find the number of relations which must exist between $K(\theta + 1)$ variables, forming K systems, each of which satisfies simultaneously equations of the orders $m, n, p\,..$ respectively; the number ϕ of these equations being anything less than θ; or ϕ being equal to θ, provided at the same time $K = mnp\,..$

Suppose $m \not< n$, $n \not< p\,..$ and write

$$[m, \theta] - \{m : m, n, p\,.. : \theta\} = N,$$
$$[n, \theta] - \{n : n, p\,.... : \theta\} = N',$$
&c.

Imagine the equations of the orders $n, p\,..$ given. Any function of the m^{th} order which is satisfied by N of the systems of values which satisfy the given equations, and any

particular equation of the m^{th} order, is satisfied by the remaining $K - N$ systems of values. Hence assuming N systems, satisfying the equations of the orders $n, p, \ldots$ but otherwise arbitrary, the remaining systems must satisfy these equations, and a completely determinate equation of the m^{th} order; *i.e.* there must be ϕ relations between the variables of each system, and consequently $\phi(K - N)$ relations in all. Similarly, if the equations of the orders $p\,.\,.$ were given, N' systems of variables might be assumed satisfying these equations, but otherwise arbitrary; the remaining $N - N'$ systems satisfy $(\phi - 1)$ determinate equations, or the number of relations between the variables is $(\phi - 1)(N - N')\,.\,.$; continuing in the same manner the total number of relations between the variables is

$$\phi(K - N) + (\phi - 1)(N - N') + (\phi - 2)(N' - N'') + \ldots.$$

in which however any term $(\phi - 1)(N - N')$ or $(\phi - 2)(N - N')$ ....&c., which becomes negative, *must be omitted.* It is obvious that we may write more simply

$$N = [m, \theta] - 1 - \{m\,;\, n, p\,.\,.\theta\},$$

$$N' = [n, \theta] - 1 - \{n\,;\, p\ldots.\theta\}, \text{ \&c.}$$

In particular, to find the relations which must exist between the coordinates of mn points in order that they may be the points of intersection of two curves of the orders m, n respectively: here $K = mn$, $N = \frac{1}{2}[m + 2]^2 - \frac{1}{2}[m - n + 2]^2 = \frac{1}{2}(2mn - n^2 + 3n)$, $N' = \frac{1}{2}(n^2 + 3n + 2)$, so that $N - N' = m(m - n) - 1$ which becomes negative when $m = n$; hence in general the required number of conditions is $mn - 3n + 1$, but when $m = n$, the number in question becomes $(n - 1)(n - 2)$.

Passing to the case of surfaces; to determine the number of relations which must exist between the coordinates of mnp points, in order that they may be the points of intersection of surfaces of the orders m, n, p respectively. The number required is

$$3(mnp - N) + 2(N - N') + (N' - N''),$$

where

$$N = [m, 3] - 1 - [m - n, 3] - [m - p, 3] + [m - n - p, 3]$$

(this last term to be omitted when $m < n + p - 3$),

$$N' = [n, 3] - 1 - [n - p, 3],$$

$$N'' = [p. 3] - 1.$$

If, for instance, $m > n + p - 3$, so as to retain the term $[m - n - p, 3]$, and $n > p$, so as to retain the term $N' - N''$, the number becomes, after all reductions,

$$2mnp + np^2 - 4np - 2p^2 - \tfrac{1}{3}(p - 1)(p - 2)(p - 3),$$

a formula given by Jacobi. If, however, $n = p$, this number must be augmented by unity. Again, for $m < n + p - 3$, the required number is

$$2mnp + np^2 - 4np - 2p^2 - \tfrac{1}{3}(p-1)(p-2)(p-3)$$
$$- \tfrac{1}{6}(n+p-m-1)(n+p-m-2)(n+p-m-3),$$

which however must be augmented by unity if $m = n$ or $n = p$, and by 3 if $m = n = p$. But without entering into further details about this part of the subject, which has been sufficiently illustrated by the examples that have been given, I pass on to notice the application of the above theory to the problem of elimination. Imagine $(\theta + 1)$ equations between the $(\theta + 1)$ variables, the first sides of these being, as before, rational and integral homogeneous functions of the variables of the orders $m, n, p \ldots$ respectively. Writing $m + n + p \ldots - \theta = r$, and multiplying the first equation by all the terms of the form $x^\alpha y^\beta \ldots$ of the degree $r - m$, the second equation by all the terms of the same form, of the degree $r - n$, and so on, there result a certain number of equations, containing all the terms $x^\alpha y^\beta \ldots$ of the degree r. But these equations are not independent; and the reasoning in the former part of the present paper shews that the number of independent equations is given by the symbol $\{r : m, n, p \ldots : \theta\}$; the number of terms $x^\alpha y^\beta \ldots$ is evidently $[r, \theta]$; and it will be shewn immediately that for the actual value of r,

$$[r, \theta] - \{r ; m, n, p \ldots : \theta\} = 0 \ldots\ldots\ldots (\text{B}).$$

So that the number of quantities to be linearly eliminated is precisely equal to the number of equations, or the elimination is always possible. I may mention also that, supposing the coefficients of all the equations to be of the order unity, the order of the result, free from extraneous factors, may be shewn to be

$$[r-m, \theta] + \ldots - 2\{[r-m-n, \theta] + \ldots\} + 3\{[r-m-n-p, \theta] + \ldots\}$$
$$- \&\text{c.} = mn \ldots + mp \ldots + np \ldots\ldots\ldots (C),$$

(the equality of which will be presently proved) a result which agrees with that deduced from the theory of symmetrical functions; but I am not in possession of any mode of directly obtaining the final result in this its most simplified form. My method, which it is not necessary to explain here more particularly, leads me to the formation of a set of functions.

$$P, Q, \ldots\ldots X, Y, Z,$$

θ in number, such that Z divides Y, this quotient divides X, and so on until we have a certain quotient which divides P,

and this quotient equated to zero is the result of the elimination freed from extraneous factors. It only remains to demonstrate the formulæ (A), (B), and (C). Suppose in general that (k) denotes the sum of all the terms of the form $m^a n^b$.., which can be formed with a given combination of k letters out of the ϕ letters m, n, p . . . And let $\Sigma\,(k)$ denote the sum of all the series (k) obtained by taking all the possible different combinations of k letters. It is evident that $\Sigma\,(k)$ is a multiple of (ϕ), [(ϕ) denoting of course the sum of all the terms $m^a n^b$. ., m, n. . being any letters whatever out of the series m, n, p. .]. Let g be the number of exponents a, b, . ., then (ϕ) contains $[\phi]^g$ terms, also (k) contains $[k]^g$ terms, and the number of terms such as (k) in the sum $\Sigma\,(k)$ is $[\phi]^{\phi-k} \div [\phi-k]^{\phi-k}$. Whence evidently

$$\Sigma\,(k) = \frac{[\phi - g]^{\phi-k}}{[\phi - k]^{\phi-k}}\,(\phi).$$

Or, what comes to the same thing,

$$\Sigma\,(\phi - k) = \frac{[\phi - g]^k}{[k]^k}\,(\phi).$$

Let A be an indeterminate coefficient, σ a summatory sign referring to different systems of exponents; then

$$\Sigma\sigma A\,(\phi - k) = \sigma\,\frac{[\phi - g]^k}{[k]^k}\,A\,(\phi).$$

Or, giving to k the values $1, 2 \ldots \phi$, multiplying each equation by an arbitrary coefficient, and adding, putting also for shortness $\sigma A\,(\phi - k) = U_{\phi-k}$, we have

$$a_\phi U_\phi + a_{\phi-1}\Sigma U_{\phi-1} + \ldots = \sigma\left(a_\phi + a_{\phi-1}\,.\,\frac{[\phi - g]^1}{[1]^1} + \ldots\right)A\,(\phi);$$

whence in particular,

$$U_\phi - \Sigma U_{\phi-1} + \ldots = \sigma\,\{0^{\phi-g}A\,(\phi)\},$$

$$\Sigma U_{\phi-1} - 2\Sigma U_{\phi-2} + \ldots = \sigma\,\{(\phi - g)\,0^{\phi-g-1}A.(\phi)\},$$

which are still equations of considerable generality. If now $\phi = \theta$ and U_θ is a function of $m + n + p + \ldots$ of the order θ, the quantity $\sigma\,\{0^{\theta-g}A\,(\theta)\}$ reduces itself to the single term of U_θ which contains the product $mnp \ldots$ Hence, if

$$U_\theta = [u + m + n + p \ldots, \theta]$$

in which afterwards $u = r - m - n - p - \ldots$ we have the formula (A). Again, if $\phi = \theta + 1$, and $U_{\theta+1}$ a function of $m + n + p \ldots$ of the order θ, the sum $\sigma\,\{0^{\theta+1-g}A\,.\,(\phi)\}$ vanishes; whence writing $U_{\theta+1} = [m + n + p \ldots - \theta, \theta]$, we have the formula (B).

Similarly, if in the second formula $\phi = \theta + 1$, and $U_{\theta+1}$ is a function of $m + n + p \ldots$ of the degree θ,

$$\sigma\{(\theta + 1 - g)\, 0^{\theta - g} A\, (\theta + 1)\},$$

reduces itself to the term which contains $mn \ldots + np \ldots + mp \ldots$; whence, if $U_{\theta+1} = [m + n + p + \ldots - \theta, \theta]$, we have the formula ($C$).

[*To be continued.*]

ON A MECHANICAL REPRESENTATION OF ELECTRIC, MAGNETIC, AND GALVANIC FORCES.

By William Thomson.

Mr. Faraday, in the eleventh series of his *Experimental Researches in Electricity*, has set forth a theory of Electro-statical Induction, which suggests the idea that there may be a problem in the theory of elastic solids corresponding to every problem connected with the distribution of electricity on conductors, or with the forces of attraction and repulsion exercised by electrified bodies. The clue to a similar representation of magnetic and galvanic forces is afforded by Mr. Faraday's recent discovery of the affection with reference to polarized light, of transparent solids subjected to magnetic or electromagnetic forces. I have thus been led to find three distinct particular solutions of the equations of equilibrium of an elastic solid, of which one expresses a state of distortion such that the absolute displacement of a particle, in any part of the solid, represents the resultant attraction at this point, produced by an electrified body; another gives a state of the solid in which each element has a certain resultant angular displacement, representing in magnitude and direction the force at this point, produced by a magnetic body; and the third represents in a similar manner the force produced by any portion of a galvanic wire; the directions of the forces in the latter cases being given by the axes of the resultant rotations impressed upon the elements of the solid.

The general equations of equilibrium of an elastic solid have been investigated by Mr. Stokes,* without the assumption of any relation between the "cubical compressibility" and the elasticity, with reference to variations of form which are not accompanied by change of volume. If we denote by

* In a paper "On the Friction of Fluids in Motion, and the Equilibrium and Motion of Elastic Solids," read at the Cambridge Philosophical Society April 14, 1845. See *Trans.*, vol. VIII. Part 3.

α, β, γ the projections on three rectangular axes of coordinates, of the infinitely small displacement of a point (x, y, z) of the solid, it follows from Mr. Stokes' results that the equations of equilibrium, when the body is acted on by no forces except at its bounding surfaces, may be written as follows:

$$\left.\begin{array}{l} -\dfrac{dp}{dx}+\dfrac{d^2\alpha}{dx^2}+\dfrac{d^2\alpha}{dy^2}+\dfrac{d^2\alpha}{dz^2}=0 \\ -\dfrac{dp}{dy}+\dfrac{d^2\beta}{dx^2}+\dfrac{d^2\beta}{dy^2}+\dfrac{d^2\beta}{dz^2}=0 \\ -\dfrac{dp}{dz}+\dfrac{d^2\gamma}{dx^2}+\dfrac{d^2\gamma}{dy^2}+\dfrac{d^2\gamma}{dz^2}=0 \\ p=-k\left(\dfrac{d\alpha}{dx}+\dfrac{d\beta}{dy}+\dfrac{d\gamma}{dz}\right) \end{array}\right\} \quad \ldots\ldots\ldots(1).$$

In the ideal limiting case in which the solid is incompressible, k will have an infinite value, and we shall have the relation

$$\frac{d\alpha}{dx}+\frac{d\beta}{dy}+\frac{d\gamma}{dz}=0 \quad \ldots\ldots\ldots\ldots (2).$$

Hence equations (1) and (2) express the conditions of the interior equilibrium of an incompressible elastic solid. These equations are to be employed for the representation of the forces in the several physical problems considered in this paper.

Now equations (1) merely shew that the expression

$$\nabla^2\alpha.dx+\nabla^2\beta.dy+\nabla^2\gamma.dz \ldots\ldots\ldots (a),$$

$\left(\text{in which } \nabla^2 \text{ denotes the operation } \dfrac{d^2}{dx^2}+\dfrac{d^2}{dy^2}+\dfrac{d^2}{dz^2}\right)$, must be a complete differential, and therefore any expressions for α, β, γ subject to this condition, which satisfy (2), will represent an interior state of the body which can be produced by the action of forces at its bounding surface or surfaces.

We may obtain a particular solution by assuming $\alpha dx + \beta dy + \gamma dz$ to be a complete differential. Again, if we suppose this expression not to be a complete differential, we may assume

$$\left(\frac{d\beta}{dz}-\frac{d\gamma}{dy}\right)dx+\left(\frac{d\gamma}{dx}-\frac{d\alpha}{dz}\right)dy+\left(\frac{d\alpha}{dy}-\frac{d\beta}{dx}\right)dz \ldots (c)$$

to be a complete differential and find another solution; or lastly we may obtain a particular solution by means of a third supposition, according to which neither of these expressions is a complete differential. These three solutions I shall now proceed to consider, with reference to the representation of Electrical, Magnetic, and Galvanic forces.

I.—*Electrical Forces.*

Let $$r^2 = x^2 + y^2 + z^2,$$

and assume $$\alpha dx + \beta dy + \gamma dz = -d\left(\frac{1}{r}\right).$$

Then, since $$\frac{d^2}{dx^2}\frac{1}{r} + \frac{d^2}{dy^2}\frac{1}{r} + \frac{d^2}{dz^2}\frac{1}{r} = 0,$$

equation (2) is satisfied, and the coefficients of the differentials in (a) vanish; so that all the conditions of equilibrium are satisfied. Now $\frac{1}{r}$ is the potential at (x, y, z), due to a unit of electricity at the origin, and

$$\alpha = \frac{x}{r^3}, \quad \beta = \frac{y}{r^3}, \quad \gamma = \frac{z}{r^3} \quad \ldots\ldots\ldots\ldots \text{(I.)}$$

are the components of the force exerted at the point (xyz).

II.—*Magnetic Forces.*

Let

$$\left(\frac{d\beta}{dz} - \frac{d\gamma}{dy}\right)dx + \left(\frac{d\gamma}{dx} - \frac{d\alpha}{dz}\right)dy + \left(\frac{d\alpha}{dy} - \frac{d\beta}{dx}\right)dz = d\,\frac{lx + my + nz}{r^3}.$$

This equation is satisfied by

$$\alpha = \frac{mz - ny}{r^3}, \quad \beta = \frac{nx - lz}{r^3}, \quad \gamma = \frac{ly - mx}{r^3} \quad \ldots\ldots (3),$$

which also satisfy equation (2), and make the coefficients of the differentials in (a) vanish. Hence, displacements expressed in this way may be produced by externally applied forces. Now

$$\frac{lx + my + nz}{r^3}$$

is the potential due to a small magnet, of which the 'moment' is unity, placed at the origin, with its axis of polarization in the direction $l:m:n$. The components X, Y, Z of the force which this magnet exerts upon an ideal unit of magnetism (one end of a thin uniformly magnetized bar) at the point x, y, z being the differential coefficients of this expression, we have

$$X = \frac{d\beta}{dz} - \frac{d\gamma}{dy}, \quad Y = \frac{d\gamma}{dx} - \frac{d\alpha}{dz}, \quad Z = \frac{d\alpha}{dy} - \frac{d\beta}{dx} \quad \ldots \text{(II.)}$$

The halves of the expressions $\frac{d\beta}{dz} - \frac{d\gamma}{dy}$, &c. indicate the

components round lines parallel to the axes, of the infinitely small rotation which an element of the solid receives, besides its change of form, when $adx + \beta dy + \gamma dz$ is not a complete differential. This rotation therefore represents the resultant magnetic force, in direction and magnitude.

III.—*Galvanic Forces.*

Let $\nabla^2 \alpha . dx + \nabla^2 \beta . dy + \nabla^2 \gamma . dz = - d \dfrac{lx + my + nz}{r^3}$,

which is true if

$$\left.\begin{aligned} \alpha &= \tfrac{1}{2} \frac{d}{dx} \frac{lx + my + nz}{r} - \frac{l}{r} \\ \beta &= \tfrac{1}{2} \frac{d}{dy} \frac{lx + my + nz}{r} - \frac{m}{r} \\ \gamma &= \tfrac{1}{2} \frac{d}{dz} \frac{lx + my + nz}{r} - \frac{n}{r} \end{aligned}\right\} \quad \ldots\ldots\ldots (4).$$

It is readily verified that these expressions also satisfy equation (2), and hence they represent an interior state of the body which may be produced by externally applied forces. Now, we find by means of these equations

$$\left.\begin{aligned} \frac{d\beta}{dz} - \frac{d\gamma}{dy} &= \frac{mz - ny}{r^3} \\ \frac{d\gamma}{dx} - \frac{d\alpha}{dz} &= \frac{nx - lz}{r^3} \\ \frac{d\alpha}{dy} - \frac{d\beta}{dx} &= \frac{ly - mx}{r^3} \end{aligned}\right\} \quad \ldots\ldots\ldots (\text{III.})$$

which are the expressions for the components of the force an infinitely small element of a galvanic current, in the direction l, m, n, at the origin, produces on a unit of magnetism at the point (x, y, z); the intensity of the current, multiplied by the length of the element, being unity. Thus we conclude that the rotation of any element of the solid, in the state expressed by (4), represents in direction and magnitude, the force of an element of a galvanic wire.

I should exceed my present limits were I to enter into a special examination of the states of a solid body representing various problems in electricity, magnetism, and galvanism, which must therefore be reserved for a future paper.

Glasgow College, Nov. 28, 1846.

ON THE DEGREE OF A SURFACE RECIPROCAL TO A GIVEN ONE.

By the Rev. GEORGE SALMON, M.A., Fellow of Trinity College, Dublin.

1. OF all the additions which modern investigation has made to the ancient geometry, none seems more important than the method of reciprocal polars, by which our knowledge of extension is at once doubled, and we are enabled from any known property of curves or surfaces at once to deduce another correlative one. Nor is it only in the multiplication of isolated theorems that this method has been useful; it has thrown new light on some important points in the general theory of curves. For instance, it is to this method that we owe an accurate investigation of the simple question, how many tangents can be drawn from a given point to a curve of the m^{th} degree? It had previously been answered, "In general $m(m-1)$." The theory of reciprocal curves made it evident that the number could not always be so great. For since the degree of the reciprocal to any curve is equal to the number of tangents that can be drawn from any point to this curve, and therefore in general $m.(m-1)$, if the degree of the reciprocal of this reciprocal were to be determined by the same rule, it would be $\{m.(m-1)\}\{m.(m-1)-1\}$, instead of m, as it plainly ought to be. It became then an interesting question, "What are the circumstances in which the number of tangents which can be drawn from a point to a curve of the m^{th} degree is less than $m.(m-1)$?"

2. The reply to this question was found to be, "When the given curve has multiple points." A little consideration shews that this result might have been anticipated; for the problem "to draw a tangent from a point to a curve," when expressed analytically, becomes "to draw a line such that two values of the radius vector may be equal;" and since this condition is satisfied by the lines drawn from the given point to the multiple points of the curve, we must deduct these lines from the $m.m-1$ solutions, of which the general question admits. It was found then that the degree of the reciprocal curve is diminished by two for every double point on the curve; by three for every cusp (that is, a double point at which the two tangents coincide); by six for every triple point, and so on.

3. As I am not aware that the corresponding question as to reciprocal surfaces has been before investigated, I purpose in the present paper to enquire, what is the degree of the surface reciprocal to one of the m^{th} degree; and to consider

how this degree is diminished if the given surface have multiple points or lines.

4. The degree of the reciprocal surface is plainly the same as the number of tangent planes which can be drawn to the surface through a given line: now we know that all the points of contact of tangent lines passing through a given point lie on a surface of the $n - 1^{\text{st}}$ degree, which we call the $(n-1)^{\text{st}}$ polar surface of that point.

In order, then, to find the points of contact of planes passing through a fixed line, we have only to take the polar surfaces of any two points on this line. The intersections of these surfaces with the given one are the points of contact required; and since three surfaces respectively of the k^{th}, l^{th}, and m^{th} degrees intersect each other in $k.l.m$ points, the number of intersections in the present case will be $m.(m-1)^2$. This therefore is the degree of the surface reciprocal to one of the m^{th} degree.

5. There is another method by which we might have determined the degree of the reciprocal surface, the result of which does not at first sight appear the same as the preceding.

The degree of the reciprocal surface is the same as that of any plane section of it; but any plane section of the reciprocal surface is reciprocal to a tangent cone of the given surface. Now the degree of the cone touching a surface of the m^{th} degree is $m.\overline{m-1}$, therefore (as the reciprocals of cones follow the same rules as curves) if the cone have no multiple sides, its reciprocal will be of the degree $m.(m-1)\{m.(m-1)-1\}$. We appear, then, to have arrived at a result contradictory to that of the preceding section.

6. I proceed to remove the apparent contradiction by establishing the following theorems. (1) "Every cone touching a surface of the m^{th} degree must in general have $\dfrac{m.(m-1)^2.\overline{m-2}}{2}$ double sides, real or imaginary." (2) "Of these double sides, $m.\overline{m-1}.\overline{m-2}$ are cuspidal lines, and consequently there are only $\dfrac{m.\overline{m-1}.\overline{m-2}.\overline{m-3}}{2}$ ordinary double lines." Assuming for a moment the truth of these theorems, we see that the degree of the curve reciprocal to this cone will be

$$m.\overline{m-1}.\left\{m^2 - m - 1 - 2.\frac{\overline{m-2}.\overline{m-3}}{2} - 3.(m-2)\right\} = m.(m-1)^2,$$

the same result at which we had arrived by the other method.

7. I proceed to prove the theorems just enunciated. It is evident that any side of the tangent cone will be a double line if it touch the surface in two different points: it is necessary, therefore, to find "How many lines can be drawn through a given point which will have double contact with a given surface?" Suppose, for greater simplicity, the point at an infinite distance, and that we enquire how many lines parallel to a given one, to the axis of z for instance, will touch the surface twice. Let $U = 0$ be the equation of the surface; $\frac{dU}{dz} = 0$ will be the equation of the surface passing through the points of contact of tangents parallel to the axis of z; and if we eliminate z between these equations, we shall have the equation of the cylinder enveloping the surface. But if any side of that cylinder touch the surface twice, the two equations $U = 0$, $\frac{dU}{dz} = 0$, which, considered as functions of z, have for every side a root in common, must for the x and y of this side have two common roots. Let us suppose then that we have solved the algebraical problem to find the two conditions necessary that these two equations considered as functions of z, should have a pair of common roots. Both of these will be functions of x and y, and will therefore represent cylinders: one of them will be the cylinder circumscribing the surface; and the intersections of this cylinder with the other will be the double sides required. All that is necessary for our present purpose is to find the degree of the second condition in x and y. Now I find that if we had two algebraical equations, one of the m^{th} the other of the n^{th} degree, the conditions that they should have two common roots will be the result of elimination between them and another equation which can be reduced to the $(m-1).(n-1)^{\text{st}}$ degree. This second equation will be in the present case of the degree $m-1 \, . \, m-2$, and the cylinder which it represents will meet the circumscribing cylinder in $\frac{m.(m-1)^2.m-2}{2}$ lines. I divide by 2 because each intersection counts for two, since from the nature of the question each is a double side of the circumscribing cylinder. The first theorem is therefore established.

8. Before proving the second, I must digress a little to state another theorem which is an immediate consequence of

the algebraical principle employed in the last section, and which is of importance in some extensions of the present investigation. "If two surfaces intersect, the projection from any point on any plane of their curve of intersection must in general have $\frac{m \,.\, n \,.\, (m-1) \,.\, (n-1)}{2}$ double points." For it is evident that the projection will have a double point whenever one of the projecting lines passes through two distinct points of the curve of intersection, and by the algebraical principle just mentioned these points will be found by combining with the equations of the given surfaces, another of the degree $\overline{m-1} \,.\, \overline{n-1}$.*

9. This property admits of a simple geometrical proof in the case where the two surfaces are of the second degree. That is to say, "The projection on any plane of the intersection of two surfaces of the second degree will in general be a curve of the fourth degree having two double points." To see this we only want to know how many lines can be drawn through a given point which will meet two surfaces of the second degree in the same pairs of points: now the point of harmonic section of such a line must be the same for both surfaces, and consequently it must meet in the same point the polar planes of the given point with regard to the two surfaces. Hence we have the following construction: "Take the polar planes of the point with regard to the two surfaces; join their intersection to the given point; the joining plane will meet the surfaces in two conics whose two common chords will pass through the given point, and these chords will meet any plane in two points which are double points on the projection of the curve of intersection."

10. I return to the proof of the second theorem stated in § 6. We know that the tangent plane at any point of a surface meets the surface in a curve of which that point is a double point (since every line drawn through that point in the tangent plane meets the surface in two indefinitely near points); and since every double point has two tangents, we learn that at any point of a surface there can be drawn two tangent lines, each of which will meet the surface in *three* indefinitely near points. If, then, from any point on one of these tangent lines we draw the enveloping cone, it will

* From this property it appears that the developable surface circumscribing two surfaces whose reciprocals are respectively of the k^{th} and l^{th} degrees will be of the degree $kl(k+l-2)$, and hence that the developable circumscribing two surfaces of the second degree will in general be of the eighth degree. I have verified this for the case of two concentric ellipsoids, but the equation is rather too long to give here.

easily appear that this line will be a cuspidal side of that cone, and the question concerning cuspidal lines reduces itself to this: Through a given point how many tangent lines can be drawn to meet the surface in three indefinitely near points?

Suppose, as before, we substitute for a cone a cylinder parallel to the axis of z, the points of such contact will be found by combining the three equations

$$U = 0, \quad \frac{dU}{dz} = 0, \quad \frac{d^2U}{dz^2} = 0,$$

which will evidently only be satisfied for $m.(m-1).(m-2)$ points.

11. Having thus shewn that the degree of the reciprocal surface is in general $m.(m-1)^2$, I proceed to examine how this degree is affected by the existence of multiple points or lines on the original surface. The case of multiple *points* presents little difficulty, as it may be treated by the same methods as for plane curves. We find, for example, by the same reasoning as for plane curves, that for every double point, nodal or conjugate, on the surface, the degree of the reciprocal surface will be diminished by 2. It may be useful to give a few illustrations of the truth of this assertion.

12. Perhaps the first case which would naturally occur to any one desirous to test the truth of such a theorem, is Fresnel's wave surface. We know that it is of the fourth degree, that its reciprocal is a similar surface, and we are accustomed to say that it has four double points. If these were all, it would appear to contradict the theory just laid down, since the reciprocal of a biquadratic surface is in general of the 36th degree, and the four double points would only reduce this number by 8.

The difficulty is solved by recollecting that we must take into account not only the real but the imaginary double points of the surface. Now the very same arguments which shew that the wave surface has four real double points in *one* of its three principal planes, prove that it has four imaginary double points in each of the other two. Moreover, the form of the equation shews that it has four other imaginary double points in the plane at infinity. Hence the wave surface has in all 16 double points, 4 real and 12 imaginary, and therefore the degree of its reciprocal $= 36 - 32 = 4$.

13. I take as another example the surface of the third degree having 4, (its maximum number) of double points.

Its equation must be of the form $A^{-1} + B^{-1} + C^{-1} + D^{-1} = 0$, $A = 0$, &c. being the equations of the four sides of the pyramid formed by the double points. This belongs to the class of surfaces $A^m + B^m + C^m + D^m = 0$, whose reciprocal is of the same form, the new m being equal to $\dfrac{m}{m-1}$. In the present case the reciprocal is of the form $A^{\frac{1}{2}} + B^{\frac{1}{2}} + C^{\frac{1}{2}} + D^{\frac{1}{2}} = 0$, a surface of the fourth degree as we expected.

14. We found in curves that though the degree of the reciprocal is only reduced by two for an ordinary double point, nodal or conjugate, yet if the tangents at it coincide, the degree will be reduced by *three*. We should expect to find an analogous result for surfaces, and accordingly I find that such is the case when the tangent cone at the double point reduces itself to two planes, real or imaginary. In this case the two surfaces of the $n - 1^{st}$ degree, whose intersections with the original we employed to determine the degree of the reciprocal, will not only pass through the double point, but will also both touch the line of intersection of the two planes; hence it appears that the degree of the reciprocal will be diminished by 3.

15. An instance of such points we have in the surface of the third degree $A.B.C = D^3$, $A = 0$, &c. being the equations of planes. Here the three points ABD, ACD, BCD are double points, and the tangent cone at any reduces to two of the planes $A = 0$, $B = 0$, $C = 0$. But the reciprocal of this surface is another of the same form, reducing from the twelfth degree to the third, on account of the three double points just mentioned.

16. Again, if the two planes coincide, both the surfaces of the $n - 1^{st}$ degree must touch this plane, and it is not difficult to see that the degree of the reciprocal surface will be reduced by 6.

Multiple points of higher degrees present no difficulty.

17. The case of multiple *lines*, however, involves much more complicated considerations. Suppose a surface to have a double line, the two polar surfaces of the $n - 1^{st}$ degree will each pass through it, and the question becomes "In how many other points will three surfaces intersect which each pass through a given curve, that curve being a double line on one of them?"

Let us commence by the simpler question: "Three surfaces respectively of the m^{th}, n^{th}, and p^{th} degrees pass through a

given right line, to how many points of intersection is this line equivalent, or in how many other points do they intersect?" The direct solution of this question is attended with some difficulty, which however we can evade by the following process.

18. We know that in general the number of points in which two curves intersect is wholly independent of the existence of multiple points on either of them (provided indeed that one of the points of intersection be not a multiple point), and that it is invariably true that a curve of the m^{th} degree cuts a curve of the n^{th} degree in mn points, real or imaginary, even when one or both curves degenerate into compound curves of lower dimensions. Following out this observation, we infer that if we could in any one case determine the number of points in which three such surfaces as we are considering intersect, we should be safe in asserting that they would always intersect in the same number of points. Let us suppose, then, that the first surface consists of a plane passing through the given line and of a surface of the $(m-1)^{\text{st}}$ dimensions. This surface will meet the other two in $m-1 \,.\, np$ points, of which $n-1$ are on the given line, and therefore the number of other points is $(m-1)\,.\,(np-1)$, and the plane meets the two surfaces in two curves of the $n-1^{\text{st}}$ and $p-1^{\text{st}}$ degrees, which intersect in $(n-1)\,.\,(p-1)$ points. The total number, therefore, of points of intersection is

$$(m-1)\,.\,(np-1)+(n-1)\,.\,(p-1)=mnp-m-n-p+2.$$

We find, then, that the common line reduces the number of points of intersection by $m+n+p-2$.

19. Let us advance now to the question: "A surface of the m^{th} degree has a double right line; two others of the degrees n and p pass through this line: in how many points not on the line will they intersect?"

Let the surface of the m^{th} degree consist of two of the degrees m' and m'' each passing through the line: from the last section we learn that the number of points of intersection is reduced by

$$m'+n+p-2+m''+n+p-2, \text{ or since } m'+m''=m,\ m+2n+2p-4.$$

Let $n=p=m-1$, and the number of points of intersection is diminished by $5m-8$. At first sight this would appear to be the number by which the degree of the reciprocal of a surface is reduced when it has a double right line. There are however some remarkable points on the double line

which affect the degree of the reciprocal, and which we have not yet taken into consideration.

20. If a surface have a double line of any kind, in general at any point of it two planes can be drawn tangent to the surface; but there will always be a determinate number of points (which I call cuspidal points), at which the two tangent planes coincide, and for each of these points the degree of the reciprocal will be further diminished by one. Take the case where the double line is a right line (suppose the axis of z), the equation of the surface will be

$$Ay^2 + Byx + Cx^2 = 0,$$

A, B, C being any functions of the variables.

The tangent planes at any point are determined by the equation $A'y^2 + B'xy + C'x^2 = 0$, where $A'B'C'$ are the values which ABC take for that point, and it is evident that these planes will coincide at the points where the axis meets the surface $B^2 = 4AC$. This surface being of the $2m - 4^{\text{th}}$ degree, we are to add this number to the number $5m - 8$ already determined, and we find that if a surface have a double right line, the degree of its reciprocal is diminished $7m - 12$. Hence the reciprocal of a surface of the third degree which has a double line, is of the third degree, since this double line is necessarily a right line.

21. If the surface have a triple right line, proceeding by a precisely similar method, we find that the degree of the reciprocal surface is diminished by $20m - 48$.

And in general, that if a surface have a multiple right line of the degree r, we find, by the same method, that the reciprocal is diminished by $(r - 1).(3r + 1)\,m - 2r.(r^2 - 1)$. Hence if a surface have a multiple line of the degree $m - 1$ (which must be a right line, since no plane can cut it in more points than one), the degree of the reciprocal will be the same as that of the given surface.

22. Let us now suppose the double line to be the curve of intersection of a surface of the k^{th} with one of the l^{th} degree.

First let us consider if three surfaces pass through such a curve, in how many other points will they intersect. Take the case where the first surface is one of the k^{th} and one of $(m - k)^{\text{th}}$; the second, one of l^{th} and of $n - l^{\text{th}}$ degree; the third, one of the p^{th}; the number of points of intersection will be

$$k.(n - l).(p - l) + l.(m - k).(p - k) + p.(m - k).(n - l),$$

therefore the general number will in this case be diminished

by $kl.\{m+n+p-(k+l)\}$. Now if one of the surfaces have this curve for a double line (suppose, as before, it to be made up of two surfaces each passing through the curve), the number of points of intersection will be diminished by

$$kl\{m'+n+p-(k+l)\}+kl\{m''+n+p-(k+l)\}$$
$$=kl.\{m+2n+2p-2(k+l)\};$$

let $n=p=m-1$, and diminution is $kl.\{5m-2.(k+l)-4\}$.

23. We must add to this the number of cuspidal points on the double line.

Let the equation of the surface be $AU^2+BUV+CV^2$, where U is of the k^{th} degree and V of the l^{th} degree. The cuspidal points are the points of intersection of $U=0$, $V=0$, and $B^2=4AC$, and are therefore in number $kl\{2m-2(k+l)\}$. Hence the degree of reciprocal is diminished by

$$kl.\{7m-4(k+l)-4\}.$$

24. We can verify this formula for the case in which the surface of the m^{th} degree consists of two, one of the k^{th} and another of the l^{th} degree. Put $m=k+l$ in the preceding, and the reciprocal is to be diminished by $kl.\{3(k+l)-4\}$; but this is precisely the difference between $(k+l).(k+l-1)^2$ and $\{k.(k-1)^2+l.(l-1)^2\}$.

25. In general let a surface of the m^{th} degree have a line of the r^{th} degree of multiplicity, said line being the intersection of a surface of the k^{th} with one of the l^{th} degree, then the degree of reciprocal will be reduced by

$$kl\{(r-1).(3r+1)m-r^2.(r-1).(k+l)-2r.(r-1)\}.$$

To verify this formula, suppose r surfaces, each of the k^{th} degree, to pass through the same curve, we must make in the above $k=l$ and $m=rk$, and the formula becomes

$$r(r^2-1)k^3-2r(r-1)k^2,$$

but this is the difference between $rk.(rk-1)^2$ and $rk.(k-1)^2$.

General as the above formula appears, it does not include some interesting cases which I am compelled to omit for the present.*

Trinity College, Dublin, Nov. 25, 1846.

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* I take this opportunity to correct a statement made by Mr. Townsend in the last number of the *Journal* (p. 36), where he ascribes to me a theorem concerning surfaces of the second degree, to the discovery of which I have no claim. It was given in the year 1836, by Professor MacCullagh, who introduced the study of these surfaces into this University, and from whom three or four years afterwards I obtained my first knowledge of the subject.

NOTE ON THE PARABOLIC POINTS OF SURFACES.

By the Rev. GEORGE SALMON.

"THE *parabolic* points on a surface of the n^{th} degree lie on the intersection of the given surface with another of the $4(n-2)$ degree." This is most easily demonstrated from the analogy which these points bear to points of inflexion on plane curves. The number of points of inflexion on a plane curve of the n^{th} degree may be found as follows, from the consideration of the well-known curves called the polar curves of any point with regard to the given curve. Let the equation of the curve be $U = 0 = u_n + \rho u_{n-1} + \rho^2 u_{n-2} + \&c.$, introducing the factor ρ to make the equation homogeneous: then the equations of the successive polar curves of the origin will be $\dfrac{dU}{d\rho} = 0$, $\dfrac{d^2u}{d\rho^2} = 0$, &c. And to come to those with which we are immediately concerned, the equation of the polar line will be $u_1 + n\rho u_0 = 0$, and of the polar conic will be

$$u_2 + (n-1)\rho u_1 + \frac{n(n-1)}{1.2}\rho^2 u_0 = 0.$$

Suppose, now, the origin to be a point of inflexion, then $u_0 = 0$ and u_1 is a factor in u_2. The last equation therefore is divisible by u_1, and hence the polar conic of a point of inflexion resolves itself into two right lines. Now if we were to find the locus of all the points whose polar conics break up into two right lines, the intersections of this locus with the original curve must include all the points of inflexion. But the equation of the polar conic of any point is

$$y^2\frac{d^2u}{dy^2} + 2xy\frac{d^2u}{dx^2dy^2} + x^2\frac{d^2u}{dx^2} + 2y\rho\frac{d^2u}{d\rho dy^2} + 2x\rho\frac{d^2u}{dx^2d\rho} + \rho^2\frac{d^2u}{d\rho^2} = 0.$$

Applying to this equation the ordinary criterion that a conic should resolve into two right lines, we obtain an equation of the third degree in $\dfrac{d^2u}{dx^2}$ &c., and consequently in the $3(n-2)$ degree in x and y. The number of points of inflexion is therefore in general $3.n.(n-2)$. But if the curve have multiple points, these points also fulfil the conditions of the question, and will therefore lie on the above-mentioned locus. The number of points of inflexion will be diminished accordingly: by six, I find, for every double point; and by eight, if the tangents at that double point coincide.

It is very easy to extend these considerations to the case of surfaces. Suppose a point on the surface to be the origin,

and the tangent plane the plane of xy, the part of the equation below the third degree will be

$$Ay^2 + Bxy + Cx^2 + z(Dy + Ex + Fz + G) = 0.$$

The equation $Ay^2 + Bxy + Cx^2 = 0$ determines the directions of the tangents at the origin to the intersection of the surface with the tangent plane; and when these directions coincide, the origin is a parabolic point. If the equation be

$$(\alpha y + \beta x)^2 + z(Dy + Ex + Fz + G) = 0,$$

it is evident that the polar of the second degree will be a cone whose vertex is the intersection of the three planes $z = 0$, $\alpha y + \beta x = 0$, $Dy + Ex + Fz + G = 0$. Hence to find the parabolic points we have only to seek the locus of a point whose polar of the second degree shall be a cone; and the intersection of this locus with the given surface will determine the points in question. But the condition that the general equation of the second degree should represent a cone, is of the fourth degree with regard to the coefficients; hence the locus required will be of the $4.n - 2$ degree.

Trinity College, Dublin, Dec. 14, 1846.

INVESTIGATION OF THE VALUE OF $\int_0^\infty \frac{\sin x dx}{x}$.*

By Francis W. Newman, Professor of Latin in University College, London.

Put $A_n = \int_0^{n\pi} \frac{\sin x dx}{x}$. Then, observing that

$$\int_{n\pi}^{(n+1)\pi} \frac{\sin x dx}{x} = \int_0^{\pi} \frac{\cos n\pi \sin x' dx'}{n\pi + x'},$$

which vanishes when n = an infinite integer, since the denominator is then infinite; it follows, *a fortiori*, that

$$\int_{n\pi}^{n\pi+\mu} \frac{\sin x dx}{x}$$

vanishes when n = an infinite integer and μ is between 0 and π: consequently we find the value of A from A_n by supposing $n = \infty$, without any difference in the result, whether n be integer or fractional.

[* Before I received this article, Mr. Cayley had communicated to me a paper containing the application of the method here given to the evaluation of various integrals, both single and double; but his results have not yet been published. The solution given in the article published at present was given by Mr. Cayley as an example.—Ed.]

Now $$\int_0^{n\pi} = \int_0^{\pi} + \int_{\pi}^{2\pi} + \int_{2\pi}^{3\pi} + \ldots + \int_{(n-1)\pi}^{n\pi}.$$

Hence we have

$$A_n = \int_0^{\pi} \frac{\sin y_1 dy_1}{y_1} + \int_{\pi}^{2\pi} \frac{\sin y_2 dy_2}{y_2} + \ldots + \int_{(n-1)\pi}^{n\pi} \frac{\sin y_n dy_n}{y_n};$$

since, when the limits are fixed, we may arbitrarily change x into y.

Let
$$y_1 = \pi - x; \quad y_2 = \pi + x;$$
$$y_3 = 3\pi - x; \quad y_4 = 3\pi + x;$$
$$y_5 = 5\pi - x; \quad y_6 = 5\pi + x; \text{ \&c....}$$

Then when $y_1, y_3, y_5, y_7 \ldots$ are at their lower limits, x is π; but at their upper limits, $x = 0$. The cases are reversed for $y_2, y_4, y_6, y_8 \ldots$. Observing, then, that $\int_{\pi}^{0} - du = \int_0^{\pi} du$, we change every term of the series into an integral, in which x varies between the same limits 0 and π; so that

$$A_n = \int_0^{\pi} \left\{ \frac{\sin xdx}{\pi - x} - \frac{\sin xdx}{\pi + x} + \frac{\sin xdx}{3\pi - x} - \frac{\sin xdx}{3\pi + x} + \frac{\sin xdx}{5\pi - x} - \text{\&c.} \ldots \text{ to } n \text{ terms} \right\}.$$

Since the denominators increase, the fractions diminish; and as they are alternately positive and negative, the sum will approximate to a single real and finite limit, when $n = \infty$, by a well-known law of infinite series. Hence, making $n = \infty$, we get

$$A = \int_0^{\pi} \left\{ \frac{1}{\pi - x} - \frac{1}{\pi + x} + \frac{1}{3\pi - x} - \frac{1}{3\pi + x} + \text{\&c. \&c.} \right\} \sin xdx.$$

But by an easy result of the equation

$$\cos \frac{x}{2} = \left(1 - \frac{x^2}{\pi^2}\right)\left(1 - \frac{x^2}{9\pi^2}\right)\left(1 - \frac{x^2}{25\pi^2}\right) \text{\&c. \&c.}$$

it is well known that the series within brackets $= \frac{1}{2} \tan \frac{x}{2}$;

$$\therefore \quad A = \int_0^{\pi} \tfrac{1}{2} \tan \frac{x}{2} \sin xdx = \int_0^{\pi} \sin^2 \frac{x}{2} dx = \int_0^{\pi} \frac{1 - \cos x}{2} dx$$

$$= \left(\frac{x}{2} - \frac{\sin x}{2}\right) \text{ within proper limits; or } A = \frac{\pi}{2}.$$

P.S. I ventured to send this Article to the Editor, at the request of a mathematician, who had doubted of the truth of the *result*, in consequence of the defectiveness of the current proof.—F. N.

ON LOGARITHMIC INTEGRALS OF THE SECOND ORDER.

By Francis W. Newman.

§. I.

1. The general formula $\int F_1x \,.\, \log F_2x \,.\, dx$, where F_1F_2 denote rational functions, contains a variety of integrals, all of which, it will be shewn, can be reduced to *three*.

By the common method of finding $\int F_1x \,.\, dx$, we perceive that there is some rational function F_3 which fulfils the equation

$$F_1x = \frac{d}{dx} F_3x + \Sigma \frac{A}{x-e} + \Sigma \frac{px+q}{(x-\mu)^2+\nu^2}.$$

Also, if F_2x be reduced to the form of a single algebraic fraction, it may be denoted by $F'x \div F''x$, where F' and F'' are each *integer*. Consequently we may write

$$\log F_2x = \Sigma . A_1 \log(ax+b) + \Sigma . A_2 \log(a'x^2+b'x+c').$$

It immediately follows that $\int F_1x \,.\, \log F_2xdx$ is separable into the two forms $\int F_1x.\log(ax+b)\,dx$ and $\int F_1x.\log(a'x^2+b'x+c')\,dx$. In the former, introduce the preceding value of F_1x, and we obtain for the integral

$$\log(ax+b) . F_3x - \int \frac{F_3x.adx}{ax+b}$$

$$+ \Sigma.A \int \frac{\log(ax+b)}{x-e} dx + \Sigma \int \log(ax+b) . \frac{(px+q)\,dx}{(x-\mu)^2+\nu^2}.$$

Of the three integrals which here appear, the first is rational. In the second assume $ax+b = mx'$; $\therefore a(x-e) = mx' - b - ae$. Assume farther, $m = b + ae$; then

$$\int \frac{\log(ax+b)\,dx}{x-e} = \log m . \int \frac{dx}{x-e} + \int \frac{\log x' . dx'}{x'-1},$$

provided that m, or $(b+ae)$, is positive. If otherwise, put $x = e + m'x''$, and $am' = -(b+ae)$;

$$\therefore \int \frac{\log(ax+b)\,dx}{x-e} = \log(ax+b) . \log \frac{x-e}{m'} - \int \frac{\log x'' . dx''}{x''-1}.$$

In either case we arrive at the elementary form

$$L(x) = \int_1 \frac{\log x.dx}{x-1} \quad \ldots\ldots\ldots\ldots \text{(1)},$$

which Spence has tabulated. As for the integral

$$\int \log(ax+b) . \frac{(px+q)\,dx}{(x-\mu)^2+\nu^2},$$

the same assumption, $ax + b = mx'$, if we give to m a suitable constant value, produces two general forms which may be denoted by

$$\int \frac{\log x\,dx}{X} \text{ and } \tfrac{1}{2}\int \log x\,d\log X\,; \text{ if } X = x^2 - 2x\cos a + 1.$$

Let ω be an arc such that $\tan\omega = x\sin a \div (1 - x\cos a)$, or, what is the same, $x = \sin\omega \div \sin(\omega + a)$: then

$$d\omega = \frac{\sin a.dx}{X}; \text{ and } \sin a.\int \frac{\log x\,dx}{X}$$

$$= \int \log\sin\omega d\omega - \int \log\sin(\omega + a)\,d\omega.$$

Suppose ϟ to be a symbol for a new function, such that

$$\text{ϟ}(\omega) = -\int_0 \log\sin\omega.d\omega \,\ldots\ldots\ldots (2);$$

then
$$\sin a.\int_0 \frac{\log x\,dx}{X} = \text{ϟ}(\omega + a) - \text{ϟ}\omega - \text{ϟ}a \ldots (2).$$

No similar reduction occurs, by which we can exterminate the arbitrary constant from the next integral; and we must be satisfied with writing

$$\Lambda(x, a) \text{ for } \int_0 \frac{\log x.(x - \cos a)\,dx}{x^2 - 2x\cos a + 1} \text{ or } \tfrac{1}{2}\int lx.dlX \ldots (3).$$

It will be sometimes convenient to put

$$\lambda(x, a) \text{ for } \tfrac{1}{2}\int_0 \log(x^2 - 2x\cos a + 1).\frac{dx}{x} \ldots (4),$$

which is a supplemental function to Λ, and so related that

$$\Lambda(x, a) + \lambda(x, a) = \tfrac{1}{2}\log x.\log X.$$

We may write Λx, λx when no change of a is contemplated.

2. We have now to go back to $\int F_1 x.\log(ax^2 + bx + c)\,dx$. By substituting as before for F_1, we reduce the integral to

$$F_3 x.\log(ax^2 + bx + c) - \int F_3 x.\frac{(2ax + b)\,dx}{ax^2 + bx + c}$$

$$+ \Sigma A\int \frac{\log(ax^2 + bx + c)}{x - e}\,dx + \Sigma\int \log(ax^2 + bx + c).\frac{(px + q)\,dx}{(x - \mu)^2 + \nu^2}.$$

Of the three integrals remaining, the first is rational. The second is readily reduced to the form λ, by making $(x - e) = mx'$. The third, by making $x - \mu = mx'$, and determining m aright, produces the two new forms

$$X_1 = \int \log X.\frac{ndx}{x^2 + n^2}; \quad X_2 = \int \log X.\frac{xdx}{x^2 + n^2};$$

each of which has two arbitrary constants, a and n. But fortunately we can reduce X_1 to ϟ, and X_2 to L or λ. First, for X_1, put $x = n \tan \omega$, $n = \tan \nu$; $\dfrac{n\,dx}{x^2 + n^2} = d\omega$.

$$X = 1 - 2n \tan \omega \cos a + n^2 \tan^2 \omega$$
$$= (\cos^2 \omega - 2n \sin \omega \cos \omega \cos a + n^2 \sin^2 \omega) \div \cos^2 \omega$$
$$= \{(1 + n^2) - 2n \sin 2\omega \cos a + (1 - n^2).\cos 2\omega\} \div 2 \cos^2 \omega$$
$$= (1 - \sin 2\nu \sin 2\omega \cos a + \cos 2\nu.\cos 2\omega) \div 2 \cos^2 \nu.\cos^2 \omega.$$

Let μ, β be taken such that $\left.\begin{array}{l}\sin \mu \sin \beta = \sin 2\nu \cos a \\ \sin \mu \cos \beta = \cos 2\nu\end{array}\right\}$;

$$\therefore \quad \cos \mu = \sin 2\nu \sin a, \text{ and } \tan \beta = \tan 2\nu.\cos a.$$

Also $X = \{1 + \sin \mu (\cos 2\omega \cos \beta - \sin 2\omega \sin \beta)\} \div 2 \cos^2 \nu.\cos^2 \omega$,

$$= (1 + \sin \mu \cos \theta) \div 2 \cos^2 \nu \cos^2 \omega; \text{ if } \theta = 2\omega + \beta:$$

whence $X_1 = \int \log X.d\omega = \frac{1}{2} \int \log (1 + \sin \mu \cos \theta)\, d\theta$

$$- \omega \log (2 \cos^2 \nu) - 2\text{ϟ}(\tfrac{1}{2}\pi - \omega). \ldots (5):$$

in which the remaining integral has but one arbitrary constant.

Farther, let $m = \tan \frac{1}{2}\mu$, or $\sin \mu = 2m \div (1 + m^2) = 2m \cos^2 \frac{1}{2}\mu$;

$\therefore \log (1 + \sin \mu \cos \theta) = \log (1 + 2m \cos \theta + m^2) + 2 \log \cos \frac{1}{2}\mu$.

Assume η such that $\tan \eta = \sin \theta \div (m + \cos \theta)$,

$$\text{or } m = \sin (\theta - \eta) \div \sin \eta,$$

$$\therefore 1 + 2m \cos \theta + m^2 = \sin^2 \theta + (m + \cos \theta)^2$$
$$= \sin^2 \theta + \left(\frac{\sin \theta}{\tan \eta}\right)^2 = \left(\frac{\sin \theta}{\sin \eta}\right)^2.$$

whence $\frac{1}{2} \int \log (1 + \sin \mu \cos \theta)\, d\theta$

$$= \int \{\log \sin \theta - \log \sin \eta + \log \cos \tfrac{1}{2}\mu\}\, d\theta$$
$$= -\text{ϟ}\theta - \int \log \sin \eta.d\theta + \theta.\log.\cos \tfrac{1}{2}\mu.$$

Now $\int \log \sin \eta.d\theta$

$$= \int \log.\frac{\sin (\theta - \eta)}{m}.d\theta = \int \log \frac{\sin (\theta - \eta)}{m}.\{d(\theta - \eta) + d\eta\}$$
$$= -\text{ϟ}(\theta - \eta) - (\theta - \eta) \log m + \int \log \sin \eta.d\eta$$
$$= -\text{ϟ}(\theta - \eta) - (\theta - \eta) \log \tan \tfrac{1}{2}\mu - \text{ϟ}\eta.$$

Therefore $\left.\begin{array}{l}\frac{1}{2} \int \log (1 + \sin \mu \cos \theta)\, d\theta \\ = \text{ϟ}(\theta - \eta) + \text{ϟ}\eta - \text{ϟ}\theta + \theta \log \sin \frac{1}{2}\mu - \eta \log \tan \frac{1}{2}\mu\end{array}\right\} \ldots (6),$

which is a general formula, provided that $\tan \eta = \dfrac{\sin \theta}{\tan \frac{1}{2}\mu + \cos \theta}$;

and completes the reduction of X_1 to the function ϟ.

3. The integral X_2 remains. Using for X the same transformation as before, let us write 2θ in place of θ, so that now $2\theta = 2\omega + \beta$. We have, moreover,

$$\frac{xdx}{x^2+n^2} = \tfrac{1}{2} d \log (n^2 + x^2) = \tfrac{1}{2} d \log \sec^2 \omega = \tan \omega d\omega$$

$$\text{and also} = -\tfrac{1}{2} d \log (2 \cos^2 \nu \,.\, \cos^2 \omega).$$

Whence

$$X_2 = \int \{\log (1 + \sin \mu \cos 2\theta) - \log (2 \cos^2 \nu \,.\, \cos^2 \omega)\} \frac{xdx}{x^2+n^2}$$

$$= \int \log (1 + \sin \mu \cos 2\theta) \tan \omega d\omega + \tfrac{1}{4} \log^2 . (2 \cos^2 \nu \,.\, \cos^2 \nu).$$

Put $b = \tan \frac{1}{2}\beta$; $t = \tan \theta$, $d\omega = d\theta = \dfrac{dt}{1+t^2}$:

$$\cos 2\theta = \frac{1-t^2}{1+t^2}; \; \tan \omega = \tan \left(\theta - \frac{b}{2}\right) = \frac{t-b}{1+bt};$$

$$\text{and } \tan \omega . d\omega = \frac{t-b}{1+bt} \cdot \frac{dt}{1+t^2} = \frac{tdt}{1+t^2} - \frac{bdt}{1+bt}.$$

Hence the integral which remains, becomes

$$\left\{\int \log \{1 + t^2 + \sin \mu \,.\, (1-t^2)\} - \log (1+t^2)\right\} . \left(\frac{tdt}{1+t^2} - \frac{bdt}{1+bt}\right).$$

Write

$T_1 = \int \log \{1 + \sin \mu + (1 - \sin \mu) \,.\, t^2\} \,.\, \frac{1}{2} d \log (1 + t^2)$,

$T_2 = \int \log (1 + t^2) \,.\, \frac{1}{2} d \log (1 + t^2) = \frac{1}{4} \log^2 (1 + t^2) = \log^2 \cos \theta$,

$T_3 = \int \log \{1 + \sin \mu + (1 - \sin \mu) \,.\, t^2\}\, d \log (1 + bt)$,

$T_4 = \int \log (1 + t^2)\, d \log (1 + bt)$.

Then $X_2 = T_1 - T_2 - T_3 + T_4 + \frac{1}{4}\log^2 . (2 \cos^2 \nu \,.\, \cos^2 \omega)$.

To find T_1, let $1 + t^2 = mv$, and $m = 2 \sin \mu \div (1 - \sin \mu)$;

$$\therefore \quad T_1 = \tfrac{1}{2} \int l \{2 \sin \mu \,.\, (1 + v)\, dl\, (mv)$$

$$= \tfrac{1}{2} l\, (2 \sin \mu)\, l\, (mv) + \tfrac{1}{2} L\, (1 + v);$$

where $mv = \sec^2 \theta$,

$$1 + v = \frac{1 + t^2 + \sin \mu\, (1 - t^2)}{2 \sin \mu} = \frac{\sec^2 \theta\, (1 + \sin \mu \cos 2\theta)}{2 \sin \mu};$$

so that $T_1 = -\log (2 \sin \mu) \log \cos \theta + \frac{1}{2} L \dfrac{1 + \sin \mu \cos 2\theta}{\sin \mu\, (1 + \cos 2\theta)}$.

For T_3, put $1 + bt = kz$, $1 + \sin \mu + (1 - \sin \mu)\, t^2$

$= b^{-2} . \{1 + b^2 - (1 - b^2) \sin \mu - 2kz\, (1 - \sin \mu) + k^2 z^2\, (1 - \sin \mu)\}$.

Take k such that $k^2 (1 - \sin \mu) = 1 + b^2 - (1 - b^2) \sin \mu$;

or $= (1 + b^2)\, \{1 - \cos \beta \sin \mu\} = \sec^2 \frac{1}{2}\beta\, (1 - \cos 2\nu)$;

$$\therefore\quad k = \sec\frac{\beta}{2}\sqrt{\frac{1-\cos 2\nu}{1-\sin\mu}}.$$

Also let $\cos\gamma = k^{-1} = \cos\dfrac{\beta}{2}\sqrt{\dfrac{1-\sin\mu}{1-\cos 2\nu}}$,

and observe that $b^{-2}k^2(1-\sin\mu) = (\sin\tfrac{1}{2}\beta)^{-2}.(1-\cos 2\nu)$,

also $kz = 1 + bt = 1 + \tan\tfrac{1}{2}\beta\tan\theta = \dfrac{\cos(\theta-\frac{1}{2}\beta)}{\cos\frac{1}{2}\beta\cos\theta} \propto \dfrac{\cos\omega}{\cos\theta}$.

Hence

$$T_3 = \int\log\{(\sin\tfrac{1}{2}\beta)^{-2}.(1-\cos 2\nu).(1-2z\cos\gamma+z^2)\}\, d\log(kz)$$

$$= \log\{(\sin\tfrac{1}{2}\beta)^{-2}.(1-\cos 2\nu)\}\log\frac{\cos\omega}{\cos\theta} + 2\lambda(z,\gamma).$$

From this we may deduce T_4 by momentarily supposing $\mu = 0$, which makes $\cos 2\nu = 0$; so that, writing $k'y$ for kz, we get $k' = \sec\frac{1}{2}\beta$, and γ changes into $\frac{1}{2}\beta$. Also $y = \dfrac{\cos\omega}{\cos\theta}$.

$$\therefore\quad T_4 = -\log\sin^2\tfrac{1}{2}\beta\log\frac{\cos\omega}{\cos\theta} + 2\lambda(y,\tfrac{1}{2}\beta),$$

and $-T_3 + T_4$

$$= -\log(1-\cos 2\nu)\log\frac{\cos\omega}{\cos\theta} + 2\lambda(y,\tfrac{1}{2}\beta) - 2\lambda(z,\gamma);$$

in which we may deduce z, γ from $y, \frac{1}{2}\beta$ by writing

$$c^2 = \frac{1-\sin\mu}{1-\cos 2\nu},\quad z = cy,\quad \cos\gamma = c.\cos\tfrac{1}{2}\beta.$$

Combining all the results, we have to observe that (neglecting constants)

$$\tfrac{1}{4}l^2(2\cos^2\nu\cos^2\omega) - l(2\sin\mu)\,l\cos\theta - l^2\cos\theta$$
$$- l(1-\cos 2\nu)(l\cos\omega - l\cos\theta)$$
$$= \log^2(n\sec\omega) - \log^2\left(\cos\theta.\frac{\sqrt{\sin\mu}}{\sin\nu}\right).$$

Whence, finally,

$$\left.\begin{aligned} X_2 &= \log^2(n\sec\omega) - \log^2\left(\cos\theta.\frac{\sqrt{\sin\mu}}{\sin\nu}\right)\\ &\quad + \tfrac{1}{2}L\frac{1+\sin\mu\cos 2\theta}{\sin\mu(1+\cos 2\theta)} + 2\lambda\left(y,\frac{\beta}{2}\right) - 2\lambda(z,\gamma)\end{aligned}\right\}\ldots(7).$$

Observe that $ny^{-1} = n\cos\frac{1}{2}\beta - x\sin\frac{1}{2}\beta$; and the quantity under L may also be denoted by $\dfrac{y^2X\cos^2\nu}{\sin\mu}$. The result thus obtained admits likewise of other forms, by means of the

properties of λ and Λ; but all that is here aimed at, is to shew the possibility of the reduction.

It is easy to verify our result, in the case of $a = \frac{1}{2}\pi$. On the whole it has appeared that the integral $\int F_1 x \log F_2 x dx$ contains only three elementary forms, which we have denoted by L, ל, Λ. It is proposed to call these Logarithmic Integrals *of the Second Order*.

4. Before leaving the integrals X_1, X_2, it may be well to examine the special cases of $n = 1$, and of $x = \infty$. First, to find X_1 when $x = \infty$.

Put $X' = \int_0 \tan^{-1} \frac{x}{n} . d \log X = \log X . \tan^{-1} \frac{x}{n} - X_1$:

$$\therefore \frac{dX'}{dn} = -\log X . \frac{x}{n^2 + x^2} - \frac{dX_1}{dn};$$

and when $x = \infty$, $\frac{dX'}{dn} = -\frac{dX_1}{dn}$.

Again, $\frac{dX'}{dn} = \int_0 \frac{-x}{n^2+x^2} . d \log X$; which we assume

$$= \int_0 \left\{ \frac{2px + 2q}{x^2+n^2} + \frac{2r(x - \cos a) + 2s \sin a}{x^2 - 2x \cos a + 1} \right\} dx;$$

and by common methods we find that if $N = n^4 + 2n^2 \cos 2a + 1$,

$$p = -r = \frac{\cos a . (n^2+1)}{N}; \; q = -\frac{n(n^2 + \cos 2a)}{N}; \; s = \frac{(n^2-1)\sin a}{N}.$$

Also $\frac{dX'}{dn} = -p \log (n^2) + p \log \frac{x^2+n^2}{X}$

$$+ 2q \tan^{-1} \frac{x}{n} + 2s . \tan^{-1} . \frac{x \sin a}{1 - x \cos a}.$$

Let $x = \infty$; then integrating for n, observing that $\frac{dX'}{dn} = -\frac{dX_1}{dn}$,

$$-X_1 = -\int_0 2 \log n . pdn + \pi \int_0 qdn + 2(\pi - a) \int_0 sdn;$$

observing that as X_1 vanishes with n, no function of x is to be added. Now

$$2pdn = \frac{2 \cos a (n^2+1) dn}{n^4 + 2n^2 \cos 2a + 1} = \frac{\cos a . dn}{n^2 - 2n \sin a + 1} + \frac{\cos a \, dn}{n^2 + 2n \sin a + 1};$$

$\therefore$ if $\tan \rho = \frac{n \cos a}{1 - n \sin a}$, and $\tan \sigma = \frac{n \cos a}{1 + n \sin a}$

$$\int_0 2 \log n . pdn = \int_0 \frac{\cos a . \log n . dn}{n^2 - 2n \sin a + 1} + \int_0 \frac{\cos a . \log n . dn}{n^2 + 2n \sin a + 1}$$

= [ל$\{\rho + (\frac{1}{2}\pi - a)\}$ − לρ − ל$(\frac{1}{2}\pi - a)$] + [ל$\{\sigma + (\frac{1}{2}\pi + a)\}$ − לσ − ל$(\frac{1}{2}\pi + a)$].

It will in a following section appear that

$$ϟ\left(\tfrac{1}{2}\pi - a\right) + ϟ\left(\tfrac{1}{2}\pi + a\right) = ϟ\pi.$$

Again, $\int_0 q\,dn = -\frac{1}{4}\log N$; $\int_0 2s\,dn = \frac{1}{2}\log . \dfrac{n^2 - 2n\sin a + 1}{n^2 + 2n\sin a + 1}$.

As before, take $\tan\beta = \tan 2\nu\cos a$, and $\cos\mu = \sin 2\nu\sin a$; add to this, $\tan\beta' = \cos 2\nu\tan a$; $\therefore \rho + \sigma = \beta$, $\rho - \sigma = a - \beta'$; from which we easily find ρ and σ. Also

$$\int_0 2s\,dn = \tfrac{1}{2}\log . \frac{1 - \sin 2\nu\sin a}{1 + \sin 2\nu\sin a} = \log\tan\tfrac{1}{2}\mu,$$

and $N = (n^2+1)^2 . (1 - \sin^2 2\nu . \sin^2 a) = \sec^4\nu . \sin^2\mu$:

so that finally,
$$\left.\begin{aligned} &\int_0^\infty \log\left(1 - 2n\tan\omega\cos a + n^2\tan^2\omega\right) d\omega \\ &= \tfrac{1}{2}\pi\log\left(\sec^2\nu . \sin\mu\right) - (\pi - a)\log\tan\tfrac{1}{2}\mu \\ &\quad + ϟ\left(\tfrac{1}{2}\pi + \rho - a\right) - ϟ\left(\tfrac{1}{2}\pi - \sigma - a\right) - ϟ\rho - ϟ\sigma \end{aligned}\right\} \ldots (8).$$

A similar process applies to X_2 when $x = \infty$; and by help of the property (to be hereafter proved) that, when $x = \infty$,

$$\left\{2\Lambda(x, a) - \log^2 x\right\} = \tfrac{2}{3}\pi^2 - 2\pi a + a^2;$$

yields
$$\left.\begin{aligned} &\left\{\log^2 x - X_2\right\} \text{ when } x = \infty, \\ &= \tfrac{2}{3}2\pi^2 - \pi a - \tfrac{1}{2}\pi\beta - (\pi - a)\beta' + \tfrac{1}{2}\Lambda\left(n^2, \pi - 2a\right) \end{aligned}\right\} \ldots (9).$$

Lastly:

When $n = 1$, $\dfrac{dX_2}{da} = \displaystyle\int_0 \frac{2x\sin a}{X} . \frac{x\,dx}{1+x^2} = \tan a . \int_0 \left\{\frac{x}{X} - \frac{x}{1+x^2}\right\} dx$

$$= \tan^{-1} . \frac{x\sin a}{1 - x\cos a} + \frac{\tan a}{2}\log\frac{X}{1+x^2}.$$

Let $\cos a = h$, and observe that

$$\frac{d\lambda(x, a)}{da} = \int_0 \frac{\sin a . dx}{X} = \tan^{-1} . \frac{x\sin a}{1 - x\cos a};$$

$$\therefore X_2 = f(x) + \lambda(x, a) - \tfrac{1}{2}\int_0 \log\left\{1 - \frac{2xh}{1+x^2}\right\}\frac{dh}{h}.$$

To find the arbitrary f, let $a = \frac{1}{2}\pi$, $h = 0$, $\therefore X_2 = \frac{1}{4}\log^2(1+x^2)$,

$$\text{and } \lambda(x, a) = \tfrac{1}{2}\int_0 \log(1+x^2)\frac{dx}{x} = \tfrac{1}{4}L(1+x^2);$$

$$\left.\begin{aligned} &\therefore f(x) = \tfrac{1}{4}\log^2(1+x^2) - \tfrac{1}{4}L(1+x^2) \\ &\text{and } X_2 = f(x) + \lambda(x, a) - \tfrac{1}{2}L\left(\frac{X}{1+x^2}\right) \end{aligned}\right\} \ldots\ldots (10),$$

which is a simpler expression than would arise from putting $n = 1$ in equation (7).

§ II.—*On Spence's Integral* $\int_1 \frac{\log x dx}{x-1}$.

5. Spence has tabulated this integral, on the assumption that x is positive; and this suffices in practice. Yet it embarrasses us in generalizing concerning the integrals which are partially reducible to L, not to be at liberty to suppose x negative. Supposing $\log x$ to have arisen out of integration, and to be $= \int \frac{dx}{x}$, no imaginary quantity results from regarding x as negative: in fact, we may look on $\log x$ as a short mode of writing $\frac{1}{2} \log x^2$; then, in passing through 0, x produces no discontinuity in L.

The following are the chief properties of L, which are easily verified:

$$Lx + L(-x) = \tfrac{1}{2} L(x^2) + \tfrac{3}{2} L0,$$

$$L(\pm x) + L(1 \mp x) = \log x \,.\, \log(1 \mp x) + L0,$$

$$Lx + Lx^{-1} = \tfrac{1}{2} \log^2 x \quad (x \text{ positive}),$$

$$L(1+x) + L(1-x) = \tfrac{1}{2} L(1-x^2),$$

$$L(1+x) + L(1+x^{-1}) = \tfrac{1}{2} \log^2 x + C;$$

where $C = 2L2$, if x is positive; but $C = 2L0$, if x is negative. This is proved by making $x = 1$ in the former case, and $x = -1$ in the latter. The discontinuity is occasioned by $L(1 + x^{-1})$ becoming infinite, when x is passing through 0. So, if we wish to make x negative in the third formula, we must add $2L(-1)$ or $-\frac{1}{2}\pi^2$ on the right-hand side. Farther, we have

$$-L0 = 2L2 = \tfrac{1}{6}\pi^2, \quad L(-1) = -3L2 = -\tfrac{1}{4}\pi^2.$$

When $(x-1)$ is infinitesimal,

$$Lx = x - 1, \quad \text{and } \tfrac{1}{4}\pi^2 + L(-x) = \left(\frac{x-1}{2}\right)^2.$$

When x is large,

$$L(-x+1) = 2L0 + \tfrac{1}{2}\log^2 x + 1^{-2}x^{-1} + 2^{-2}x^{-2} + 3^{-2}x^{-3} + 4^{-2}x^{-4} + \&c. \ldots$$

If we desire to know $L(-x)$ numerically, we may either calculate it by the last formula, or (when x is not large) deduce it by the first or second of the equations from Spence's Table.

In future I shall always employ $\log x$ as a mere representation of $\int \frac{dx}{x}$ or $\frac{1}{2}\log(x^2)$; and it will only be necessary,

in correcting integrals, to observe whether the arbitrary constant is altered by supposing the quantity under *log* to pass from positive to negative.

§. III.—*On the integral*, $-\int_0 \log \sin x\, dx$.

6. Since $\log \sin x$ and $\log \sin (-x)$ are by hypothesis the same, or to speak otherwise, since $\text{ל}(x) = -\frac{1}{2}\int_0 \log \sin^2 x\, dx$,

$$\therefore \quad \text{ל}(-x) = -\text{ל}x. \ldots\ldots\ldots\ldots (11).$$

Also $\text{ל}(n\pi \pm x) = \mp \int \log \sin (n\pi \pm x)\, dx = \mp \int \log \sin x\, dx$,

$$\text{or} \quad \text{ל}(n\pi \pm x) = \text{ל}(n\pi) \pm \text{ל}x.$$

Make n successively $1, 2, 3, \ldots$ and $x = \pi$ } and we find $\text{ל}(n\pi) = n\text{ל}\pi$.

Hence it readily follows that

$$\left.\begin{array}{r} \text{ל}(n\pi \pm x) = n\text{ל}\pi \pm \text{ל}x \\ \text{ל}(\pi - x) = \text{ל}\pi - \text{ל}x; \quad 2\text{ל}\,\tfrac{1}{2}\pi = \text{ל}\pi \end{array}\right\} \ldots\ldots (12).$$

These equations indicate, that to tabulate ל from $x = 0$ to $x = \frac{1}{2}\pi$ will suffice.

7. To find $\text{ל}\pi$.

Since $-\log (2 \sin x)$

$$= \cos 2x + 2^{-1} \cos 4x + 3^{-1} \cos 6x + 4^{-1} \cos 8x + \&c.,$$

therefore $\text{ל}x = x \log 2$

$$+ \tfrac{1}{2}\{1^{-2} \sin 2x + 2^{-2} \sin 4x + 3^{-2} \sin 6x + \&c.\ldots\} \ldots (13).$$

Hence $\quad \text{ל}\pi = \pi \log 2 = 2{\cdot}177586\ 0933046.$

Also $\quad \text{ל}\,\tfrac{1}{4}\pi = \tfrac{1}{4}\text{ל}\pi + \tfrac{1}{2}\{1^{-2} - 3^{-2} + 5^{-2} - 7^{-2} + 9^{-2} - \&c.\}.$

8. Since $\sin 2x = 2 \sin x \sin (\frac{1}{2}\pi - x)$, take logs. and integrate;

$$\therefore \quad \tfrac{1}{2}\text{ל}(2x) = (\tfrac{1}{2}\pi - x) \log 2 + \text{ל}x - \text{ל}(\tfrac{1}{2}\pi - x) \ldots (14).$$

We may generalize this theorem. Since

$$\sin nx = 2^{n-1}.\sin x.\sin\left(\frac{\pi}{n} + x\right).\sin\left(\frac{2\pi}{n} + x\right)\ldots\sin\left(\frac{n-1}{n}\pi + x\right),$$

take the logarithms, as before, and integrate;

$$\therefore \frac{1}{n}\text{ל}(nx) = C - (n-1)\,x \log 2 + \text{ל}x + \text{ל}\left(\frac{\pi}{n} + x\right) + \ldots$$
$$+ \text{ל}\left(\frac{n-1}{n}\pi + x\right):$$

To find C, make $x = 0$;

$$\therefore \quad -C = \text{ל}\frac{\pi}{n} + \text{ל}\frac{2\pi}{n} + \ldots + \text{ל}\frac{n-1}{n}\pi.$$

In inverted order,

$$-C = \daleth\frac{n-1}{n}\pi + \daleth\frac{n-2}{n}\pi + \ldots + \daleth\frac{\pi}{n}.$$

Add these together, observing that

$$\daleth\left(\frac{r\pi}{n}\right) + \daleth\left(\frac{n-r}{n}\pi\right) = \daleth\pi;$$

$$\therefore\ -2C = (n-1)\daleth\pi = (n-1)\pi\log 2,$$

whence $\frac{1}{n}\daleth(nx) = -(n-1)(\tfrac{1}{2}\pi + x)\,l2$

$$+\daleth x + \daleth\left(\frac{\pi}{n} + x\right) + \ldots + \daleth\left(\frac{n-1}{n}\pi + x\right) \ldots (15).$$

If we change x to $-x$, remembering (11),

$$\frac{1}{n}\daleth(nx) = +(n-1)(\tfrac{1}{2}\pi - x)\,l2 + \daleth x - \daleth\left(\frac{\pi}{n} - x\right) - \&c.\ldots$$
$$-\daleth\left(\frac{n-1}{n}\pi - x\right),$$

which contains (14) as a particular case.

From either of them, by help of (12), putting $n = 3$ and $n = 5$,

$$\left.\begin{aligned}\tfrac{1}{3}\daleth(3x) &= -2xl2 + \daleth x + \daleth(60^\circ + x) - \daleth(60^\circ - x)\\ \tfrac{1}{5}\daleth(5x) &= -4xl2 + \daleth x + \daleth(36^\circ + x) - \daleth(36^\circ - x)\\ &\quad + \daleth(72^\circ + x) - \daleth(72^\circ - x)\end{aligned}\right\}\ldots(16).$$

If in (14) we make $x = 30^\circ$, and in the former equation of (16) make $x = 15^\circ$, we get, by help of (14),

$$\left.\begin{aligned}\tfrac{3}{2}\daleth\, 60^\circ &= \daleth\, 30^\circ + \tfrac{1}{3}\daleth\pi\\ \tfrac{4}{3}\daleth\, 45^\circ &= 2\daleth\, 15^\circ - \tfrac{1}{2}\daleth\, 30^\circ + \tfrac{1}{4}\daleth\pi\end{aligned}\right\}\ldots\ldots(17).$$

9. By help of equation (14), if a table of $\daleth$ has been computed from $x = 0$ to $x = 45^\circ$, we can continue it to $x = 90^\circ$.

Generally, if the table be given from $x = 0$ to $x = a$, we can work by a double process, as follows. First, suppose $2x$ to vary from 0 to a, in which case $\daleth x$ and $\daleth(2x)$ being known, we determine $\daleth(90^\circ - x)$ by equation (14): thus $\daleth x$ becomes known from $x = 90^\circ$ to $x = 90^\circ - \tfrac{1}{2}a$.

Next, let $2x$ vary within the last-named limits, *supposing a to be not less than* 45°, and x may lie within the limits $x = 0$, $x = a$; thus $\daleth(2x)\ \daleth(x)$ are again known; and we deduce $\daleth(\tfrac{1}{2}\pi - x)$; that is, we find $\daleth x$ from $x = \tfrac{1}{4}\pi$ to $x = \tfrac{1}{4}(\pi + a)$. Let $a_2 = \tfrac{1}{4}(\pi + a)$; and the process may be repeated, writing a_2 for a; then we fill the table as *high* as $x = \tfrac{1}{4}(\pi + a_2) = a_3$, and

as *low* as $x = \frac{1}{2}(\pi - a_2)$. Again, let $a_4 = \frac{1}{4}(\pi + a_3)$; and, by a third process, we rise as high as $x = a_4$ and come down as low as $x = \frac{1}{2}(\pi - a_3)$; and so on.

Now $a_3 = 4^{-1}\pi + 4^{-2}(\pi + a)$; $a_4 = 4^{-1}\pi + 4^{-2}\pi + 4^{-3}(\pi + a)$; &c.

Ultimately $a_\infty = \pi\,\{4^{-1} + 4^{-2} + 4^{-3} + 4^{-4} + \ldots\} = \frac{1}{3}\pi$;

and $\quad \frac{1}{2}(\pi - a_\infty) = \frac{1}{2}(\pi - \frac{1}{3}\pi) = \frac{1}{3}\pi$.

Thus the opposite series meet, and the table is filled.

In practice, if x in the table passes from degree to degree, the steps will be as follows. Given the table up to $x = 45^\circ$.

First; let $x = 1^\circ, 2^\circ \ldots 22^\circ$; and, by (14), fill the table from $x = 89^\circ$ to $x = 68^\circ$.

Next; let $x = 34^\circ, 35^\circ \ldots 44^\circ$; and fill from 56° to 46°.

Thirdly; let $x = 23^\circ, 24^\circ \ldots 28^\circ$; and fill from 67° to 62°.

Fourthly; let $x = 31^\circ, 32^\circ, 33^\circ$; and fill from 59° to 57°.

Fifthly; let $x = 29^\circ$, and we get $\daleth\,61^\circ$.

Finally; $\daleth\,60^\circ$ is found from $\daleth\,30^\circ$ by (17).

10. If we combine the use of (14) with the former of equations (16), we can fill the whole table by starting from the limit $x = 30^\circ$: and although errors might accumulate in so long a process, equations (15), (16) give us such easy modes of verification, that this perhaps is not to be feared. From $x = 0$ to $x = 30^\circ$, $\daleth x$ may be found with two or three decimal figures more than are wanted in the higher parts of the table, which will obviate this difficulty.

To give conciseness to the following explanation, write y for x in the former of equations (16), then we have

$$(a)\quad \begin{cases} \frac{1}{2}\daleth(2x) = (\frac{1}{2}\pi - x)\,l2 + \daleth x - \daleth(90^\circ - x), \\ \frac{1}{3}\daleth(3y) = -\,2y\,l2 + \daleth y + \daleth(60^\circ + y) - \daleth(60^\circ - y). \end{cases}$$

Suppose $\daleth x$ to have been found as high as $x = 30^\circ$. Find

$$\daleth\,60^\circ \text{ and } \daleth\,45^\circ \text{ by (17).}$$

Put $x = 1^\circ, 2^\circ, \ldots 15^\circ$, and find $\daleth x'$ from $x' = 89^\circ$ to $x' = 75^\circ$.
In equations (a) make $x = 18^\circ$, $y = 24^\circ$;

$$\therefore\quad \left.\begin{aligned} \tfrac{1}{2}\daleth\,36^\circ + \daleth\,72^\circ &= \tfrac{2}{5}\daleth\pi + \daleth\,18^\circ \\ \daleth\,36^\circ + \tfrac{1}{3}\daleth\,72^\circ &= \daleth\,24^\circ + \daleth\,84^\circ - \tfrac{4}{15}\daleth\pi \end{aligned}\right\}.$$

Since the right-hand members of these two equations are known, we can solve for $\daleth\,36^\circ$ and $\daleth\,72^\circ$.

Put $y = 29^\circ, 28^\circ, \ldots 25^\circ$; then $\daleth y$, $\daleth(3y)$ and $\daleth(60^\circ + y)$ being known, we can deduce $\daleth(60^\circ - y)$; *i.e.* we find $\daleth y'$ from $y' = 31^\circ$ to $y' = 35^\circ$.

Make $y=20°$, and we find $ϟ 40°$: make $x=36°$, and we find $ϟ 54°$.
.... $x=27°$, ...$ϟ 63°$: .... $y=21°$, ...$ϟ 39°$.
.... $y=18°$, ...$ϟ 42°$: .... $x=21°$, ...$ϟ 69°$.
.... $y=23°$, ...$ϟ 37°$: .... $y=9°$, ...$ϟ 51°$.
.... $y=17°$, ...$ϟ 43°$: .... $y=12°$, ...$ϟ 48°$.
.... $y=16°$, ...$ϟ 44°$: .... $x=24°$, ...$ϟ 66°$.
.... $y=22°$, ...$ϟ 38°$: .... $x=33°$, ...$ϟ 57°$.
.... $y=19°$, ...$ϟ 41°$.

Thus the table is filled as high as $x = 45°$; and the gaps in the upper portion of it may be completed by the former method.

11. To expand $ϟ x$ in converging series, when x does not exceed 30°.

First, put $\sin x = y$,

$$\therefore \quad ϟ x = -x \log y + \int \sin^{-1} y \,.\, y^{-1} dy.$$

Expand $\sin^{-1} y$ and integrate. There results

$$ϟ x = -x \log x + 1^{-2} \sin x + \tfrac{1}{2} \,.\, 3^{-2} \sin^3 x + \frac{1.3}{2.4} \,.\, 5^{-2} \sin^5 x + \&c. \ldots (18).$$

Thus, in particular, if $x = \frac{1}{6}\pi$,

$$ϟ 30° = \tfrac{1}{6} ϟ \pi + 1^{-2} . 2^{-1} + \tfrac{1}{2} . 3^{-2} . 2^{-3} + \frac{1.3}{2.4} 5^{-2} . 2^{-5} + \&c. \ldots$$

Next, let $S_n = 1^{-n} + 2^{-n} + 3^{-n} + \&c. \ldots$ a known sum; and $x = \pi\omega$;

$$\therefore \quad \log \sin (\pi\omega) = \log (\pi\omega) - S_2 \frac{\omega^2}{1} - S_4 \frac{\omega^4}{2} - S_6 \frac{\omega^6}{3} - \&c. \ldots$$

Integrate:

$$\frac{1}{\pi} ϟ (\pi\omega) = \omega \left\{1 - \log \pi\omega\right\} + S_2 \,.\, \frac{\omega^3}{1.3} + S_4 \,.\, \frac{\omega^5}{2.5} + S_6 \,.\, \frac{\omega^7}{3.7} + \&c. \quad \ldots\ldots (19).$$

To increase the convergence, add to the penultimate series before integration:

$$-\log (1 - \omega^2) = \omega^2 + \tfrac{1}{2}\omega^4 + \tfrac{1}{3}\omega^6 + \ldots \&c.$$

$$\therefore \quad -\log \sin (\pi\omega) + \log (1 - \omega^2)$$

$$= -\log (\pi\omega) + (S_2 - 1) \frac{\omega^2}{1} + (S_4 - 1) \tfrac{1}{2}\omega^4 + (S_6 - 1) \tfrac{1}{3}\omega^6 + \&c.$$

whence

$$\left.\begin{aligned} \frac{1}{\pi} ϟ (\pi\omega) = \omega\{3 - \log \pi\omega - \log(1-\omega) - \log(1+\omega)\} - \log \frac{1+\omega}{1-\omega} \\ + (S_2 - 1) \frac{\omega^3}{1.3} + (S_4 - 1) \frac{\omega^5}{2.5} + (S_6 - 1) \frac{\omega^7}{3.7} + \&c. \ldots \end{aligned}\right\} \ldots (20),$$

and if ω is less than $\frac{1}{6}$, each term of the series is less than 144th of that which precedes it.

If the coefficients are formed into a table, and the series be adapted (if necessary) to the common logarithms, it will enable us to compute $\daleth x$ from $x = 0$ to $x = 30^\circ$ with much ease. The most troublesome part of the calculation, when many decimal places are required, is the multiplying by π, (or by π . hyp. log. 10, as the case may need.)

12. We may modify the process so as to obtain a somewhat simpler series, thus: Since

$$\daleth x = -x \log \sin x + \int_0 x \cot x \, dx,$$

also $$\cot x = \frac{1}{x} - \frac{2x}{\pi^2 - x^2} - \frac{2x}{4\pi^2 - x^2} - \frac{2x}{9\pi^2 - x^2} - \&c.,$$

and $$\int_0 \frac{-2x^2 \, dx}{r^2\pi^2 - x^2} = 2x - r\pi \,.\, \log \frac{r\pi + x}{r\pi - x}; \quad \text{let } x = \pi\omega:$$

$$\left.\begin{aligned} \therefore \frac{1}{\pi}\daleth(\pi\omega) &= \omega\,(1 - l \sin.\,\pi\omega) \\ &+ \left(2\omega - l\,.\,\frac{1+\omega}{1-\omega}\right) + \left(2\omega - 2l\,\frac{2+\omega}{2-\omega}\right) + \ldots . \\ &\ldots . + \left(2\omega - r \log.\,\frac{r+\omega}{r-\omega}\right) + \ldots \end{aligned}\right\} \ldots (21).$$

Take n terms of this series, so that $r = (n - 1)$ in the last, and let R equal the remainder. Put N_m for $n^{-m} + (n+1)^{-m} + (n+2)^{-m} + \&c.$ . . and observe that

$$2\omega - r \log \frac{r+\omega}{r-\omega} = -2\omega \left\{\frac{\omega^2}{3r^2} + \frac{\omega^4}{5r^4} + \frac{\omega^6}{7r^6} + \&c.\right\};$$

$$\therefore \tfrac{1}{2}R = -N_2 \tfrac{1}{3}\omega^3 - N_4 \tfrac{1}{5}\omega^5 - N_6 \,.\, \tfrac{1}{7}\omega^7 - \&c. \ldots (22).$$

If we take the most obvious case of $n = 2$, $N_m = S_m - 1$;

$$\left.\begin{aligned} \therefore \frac{1}{\pi}\daleth(\pi\omega) &= 3\omega - \omega \log \sin.\,\pi\omega - \log \frac{1+\omega}{1-\omega} \\ -(S_2 - 1)\tfrac{1}{3}2\omega^3 &- (S_4 - 1)\tfrac{1}{5}2\omega^5 - (S_6 - 1)\tfrac{1}{7}2\omega^7 - \&c. \end{aligned}\right\} \ldots (23).$$

If between the two formulas (20) and (23) we eliminate the term which contains $(S_2 - 1)$, we get

$$\frac{3}{\pi}\daleth(\pi\omega) = \omega\,\{9 - 2\log \pi\omega - \log \sin \pi\omega - 2 \log.(1 - \omega^2)\}$$

$$-3 \log \frac{1+\omega}{1-\omega} - (S_4 - 1)(1 - \tfrac{1}{2})\tfrac{1}{5}2\omega^5 - (S_6 - 1)(1 - \tfrac{1}{3})\,.\,\tfrac{1}{7}2\omega^7 - \&c. . .$$

which may have some advantage when $(S_4 - 1) \,.\, \frac{1}{5}\omega^5$ is small enough to omit.

13. To expand $\daleth(90^\circ - x)$ when x is small.

$\daleth(\frac{1}{2}\pi - x) = \int \log \cos x \, dx = \daleth \frac{1}{2}\pi + x \log \cos x + \int_0 \tan x \, . \, x dx.$

Let $\tan x = z$, and assume

$$\frac{z \tan^{-1} z}{1 + z^2} = A_1 z^2 + A_2 z^4 + A_3 z^6 - \&c.\,;$$

$$\therefore \quad \tan^{-1} z = \left.\begin{matrix} A_1 z - A_2 z^3 + A_3 z^5 - A_4 z^7 + \&c.\,.\,. \\ + A_1 z^3 - A_2 z^5 + A_3 z^7 - \&c.\,.\,. \end{matrix}\right\}$$

whence $\quad A_1 = 1, \; A_2 = 1 + \frac{1}{3}, \; A_3 = 1 + \frac{1}{3} + \frac{1}{7}$, &c...

also $\displaystyle\int_0 \tan x \, . \, x dx = \int_0 \frac{z \tan^{-1} z \, . \, dz}{1 + z^2} = A_1 \tfrac{1}{3} z^3 - A_3 . \tfrac{1}{5} z^5 + A_3 \tfrac{1}{7} z^7 - \&c.\,.\,.$

$$= \left\{\tfrac{1}{3} z^3 - \tfrac{1}{5} z^5 + \tfrac{1}{7} z^7 - \&c.\,.\,.\right\} - \tfrac{1}{3}\left\{\tfrac{1}{5} z^5 - \tfrac{1}{7} z^7 + \tfrac{1}{9} z^9 - \&c.\right\}$$
$$+ \tfrac{1}{5}\left\{\tfrac{1}{7} z^7 - \tfrac{1}{9} z^9 + \&c.\,.\,.\right\} - \tfrac{1}{7} \;\&c.\;\&c.\,.\,.\,.$$

Let us henceforth use $\phi_n x$ for $\int_0 \tan^{n-1} x \, . \, dx$,

$$\text{or} \quad \phi_n x = \frac{\tan^n x}{n} - \frac{\tan^{n+2} x}{n + 2} + \frac{\tan^{n+4} x}{n + 4} - \&c.\,.\,.\,.$$

so that $\phi_1 x = x$; $\phi_2 x = \log \sec x$; and $\phi_{n+2} x = \dfrac{\tan^n x}{n} - \phi_n x$:

and we finally obtain

$$\daleth(\tfrac{1}{2}\pi - x) = \daleth \tfrac{1}{2}\pi + x \log \cos x$$
$$+ \phi_3 x - \tfrac{1}{3}\phi_5 x + \tfrac{1}{5}\phi_7 x - \tfrac{1}{7}\phi_9 x + \&c.\,.\,.\,.\,(24),$$

When x is $< 10^\circ$, $\phi_7 x$ will not affect the sixth decimal.

To obtain a more converging series, let $v = 1 - \cos x$.

$$\int_0 \tan x \, . \, x dx = \int - x \, . \, d \log \cos x = \int_0 \frac{\cos^{-1}(1 - v) \, . \, dv}{1 - v}.$$

Now $\cos^{-1}(1 - v) = \sqrt{(2v)}\left\{1 + \dfrac{1}{2.3} \, . \, (\tfrac{1}{2}v) + \dfrac{1.3}{2.4.5} \, . \, (\tfrac{1}{2}v)^2 + \&c.\right\}$.

Let, then, $\dfrac{\cos^{-1}(1 - v)}{1 - v} = \sqrt{(2v)}\,\{B_1 + B_2 v + B_3 v^2 + \&c.\,.\,.\,.\}$

and we get $B_1 = 1$, $B_2 = B_1 + \frac{1}{2} . \frac{1}{3} 2^{-1}$; $B_3 = B_2 + \dfrac{1.3}{2.4} . \frac{1}{5} 2^{-2}$; &c...

whence $B_\infty = 1 + \frac{1}{2} . \frac{1}{3} 2^{-1} + \dfrac{1.3}{2.4} . \frac{1}{5} 2^{-2} + \dfrac{1.3.5}{2.4.6} . \frac{1}{7} 2^{-3} + \&c.\,.\,.$

$$= \frac{1}{\sqrt{2}} \cos^{-1} . \{1 - 1\} = \frac{\pi}{2\sqrt{2}}.$$

To increase therefore the convergence, put $C_n = \dfrac{\pi}{2\sqrt{2}} - B_n$;

so that $\qquad C_{n+1} - C_n = B_n - B_{n+1}$;

$$\therefore \ \frac{\cos^{-1}(1-v)}{1-v} = \frac{\pi}{2\sqrt{2}} \cdot \sqrt{(2v)}\,(1+v+v^2+v^3+..)$$

$$- \sqrt{(2v)}\,(C_1 + C_2 v + C_3 v^2 + C_4 v^3 + \ldots .)$$

$$= \tfrac{1}{2}\pi \cdot \frac{\sqrt{v}}{1-v} - \sqrt{2}\,(C_1 v^{\frac{1}{2}} + C_2 v^{\frac{3}{2}} + C_3 v^{\frac{5}{2}} + \&c.. \ldots),$$

whence $\displaystyle\int_0 \frac{\cos^{-1}(1-v)}{1-v}\,dv = \tfrac{1}{2}\pi \log \frac{1+\sqrt{v}}{1-\sqrt{v}} - \pi\sqrt{v}$

$$- 2\sqrt{2}\,\{C_1 \tfrac{1}{3} v^{\frac{3}{2}} + C_2 \tfrac{1}{5} v^{\frac{5}{2}} + \&c.. \ldots\}.$$

Put $C_0 = \dfrac{\pi}{2\sqrt{2}}$ for uniformity;

$$\therefore \ C_1 = C_0 - 1\,; \ C_2 = C_1 - \tfrac{1}{2} \cdot \tfrac{1}{3} 2^{-1}\,; \ C_3 = C_2 - \frac{1.3}{2.4} \cdot \tfrac{1}{5} 2^{-2}\,; \ \&c. \ldots$$

and C_0, C_1, C_2.. may be easily tabulated. Finally, observing that $\sqrt{(2v)} = 2 \sin \frac{1}{2}x$, and modifying the fraction under *log*

$$\daleth(\tfrac{1}{2}\pi - x) = \daleth\tfrac{1}{2}\pi + x \log \cos x + \tfrac{1}{2}\pi \log \frac{\tan \frac{1}{4}(\frac{1}{2}\pi + x)}{\tan \frac{1}{4}(\frac{1}{2}\pi - x)}$$

$$- 4 \sin \tfrac{1}{2}x\,\{C_0 + C_1 \cdot \tfrac{1}{3}v + C_2 \cdot \tfrac{1}{5}v^2 + C_3 \cdot \tfrac{1}{7}v^3 + \&c..\}..(25)$$

which converges well even when x is as large as 60°.

14. In constructing a table of $\daleth x$ we need to find $\Delta\daleth x$, or $\daleth(x+h) - \daleth x$.

Taylor's theorem may of course be used, but the law of the terms is cumbrous. As a substitute for it, let us recal equation (2), which gave

$$\sin a \cdot \int_0 \frac{\log x\,dx}{X} = \daleth(\omega + a) - \daleth\omega - \daleth a, \text{ where } \tan \omega = \frac{x \sin a}{1 - x \cos a}.$$

whence

$$\omega = \tan^{-1}\left(\frac{x \sin a}{1 - x \cos a}\right) = x \sin a + \tfrac{1}{2}x^2 \sin 2a + \tfrac{1}{3}x^3 \sin 3a + ..$$

Also $\displaystyle \sin a \cdot \int_0 \frac{\log x\,dx}{X} = \int_0 \log x \cdot d\omega = \omega \log x - \int_0 \omega \cdot \frac{dx}{x}\,;$

$$\therefore \daleth(\omega + a) - \daleth a = \daleth\omega + \omega \log x$$

$$- 1^{-2}x \sin a - 2^{-2}x^2 \sin 2a - 3^{-2}x^3 \sin 3a - \&c.$$

To conform to the usual notation, write x, h for a, ω, and then we must put y for x. We hereby find that

If y stands for $\dfrac{\sin h}{\sin(x+h)}$,

$$\therefore \ \Delta\daleth x = \daleth h + h \log y$$

$$- 1^{-2}y \sin x - 2^{-2}y^2 \sin 2x - 3^{-2}y^3 \sin 3x - \&c..(26).$$

If x is large compared with h, and $\daleth h$ is known, y will be small enough to give a good convergence, and the law of the series is simple and convenient. Nevertheless, methods of interpolation, such as the following, will be often better.

First, let $m_1, m_2, m_3 \ldots$ satisfy the equation

$$x \div \log(1+x) = 1 + m_1 x + m_2 x^2 + m_3 x^3 + \ldots .$$

then $\Delta \int Fx\, dx = h\,\{Fx + m_1 \Delta Fx + m_2 \Delta^2 Fx + \&c. \ldots .\}$

and by slightly modifying the process, it is easy to shew that we have also

$$\Delta \int Fx\, dx = h\,\{F(x+h) - m_1 \Delta Fx + m_2 \Delta^2 F(x-h) - m_3 \Delta^3 F(x-2h) + \&c.\},$$

where F is any function whatever. Here we assume

$$Fx = -\log \sin x :$$

observe that $\quad m_1 = \frac{1}{2},\ m_2 = -\frac{1}{12},\ m_3 = \frac{1}{24};$

then, nearly,

$$\Delta \daleth x = -h\,\{l \sin x + \tfrac{1}{2}\Delta l \sin x - \tfrac{1}{12}\Delta^2 l \sin x + \tfrac{1}{24}\Delta^3 l \sin x\} \,.\,.(27),$$

$$\text{or } \Delta \daleth x = -h\,\{l \sin(x+h) - \tfrac{1}{2}\Delta l \sin x - \tfrac{1}{12}\Delta^2 l \sin(x-h) - \tfrac{1}{24}\Delta^3 l \sin(x-2h)\} \quad .(28).$$

Take the sum:

$$\therefore\ 2\Delta \daleth x = -h\,[l \sin(x+h) - \tfrac{1}{12}\{\Delta^2 l \sin x + \Delta^2 l \,.\, \sin(x-h)\} + l \sin x + \tfrac{1}{24}\{\Delta^3 l \sin x - \Delta^3 l \sin(x-2h)\}];$$

or, when the last term is negligible,

$$\Delta \daleth x = -\tfrac{1}{2}h\,[l \sin(x+h) + l \sin x - \tfrac{1}{12}\{\Delta^2 l \sin x + \Delta^2 l \sin .\,(x-h)\}] \,.\,.(29).$$

But perhaps certain series of Legendre's are better still.

Let $M_1, M_2, M_3 \ldots$ be determined by the equation

$$\tfrac{1}{2}x \div \sin^{-1}(\tfrac{1}{2}x) = 1 + M_1 x^2 + M_2 x^4 + M_3 x^6 + \ldots$$

then $\quad M_1 = \dfrac{1}{24},\quad M_2 = -\dfrac{17}{24^2 .\, 10},\quad M_3 = \dfrac{367}{8^2 .\, 5.6.7.8.9};$

and we have

$$\Delta \int Fx\, dx = h\,\{F(x+\tfrac{1}{2}h) + M_1 \Delta^2 F(x-\tfrac{1}{2}h) + M_2 \Delta^4 F(x-\tfrac{1}{2}3h) + \&c. \ldots\}$$

$$\text{also} \quad = h\,\{F(x+\tfrac{1}{2}h) + M_1 \Delta^2 F(x+\tfrac{1}{2}3h) + M_2 \Delta^4 F(x+\tfrac{1}{2}5h) + \&c. \ldots\}$$

which here give

$$\left.\begin{aligned} \Delta \daleth x &= -h\,\{l \sin(x+\tfrac{1}{2}h) \\ &\quad + \tfrac{1}{24}\Delta^2 l \sin(x-\tfrac{1}{2}h) - \tfrac{17}{5760}\Delta^4 l \sin(x-\tfrac{1}{2}3h) + \&c.\} \\ \text{also } &= -h\,\{l \sin(x+\tfrac{1}{2}h) \\ &\quad + \tfrac{1}{24}\Delta^2 l \sin(x+\tfrac{1}{2}3h) - \tfrac{17}{5760}\Delta^4 l \sin(x+\tfrac{1}{2}5h) + \&c.\} \end{aligned}\right\} \,.\,.(30),$$

which are easy to us, because we have tables of *log sin*.

Again, let N_1, N_2, N_3. . . .be such that

$$1 - N_1x^2 + N_2x^4 - N_3x^6 + \&c.\ldots = (1 + M_1x^2 + M_2x^4 + \&c.\ldots)^2;$$

$$\therefore \Delta^2 \int Fxdx = h^2 \left\{F'(x + h) + N_1\Delta^2 F'x + N_2\Delta^4 F'(x - h) + \&c.\right\}$$

$$\text{also} = h^2 \left\{F'(x + h + N_1\Delta^2 F'(x + 2h) + N_2\Delta^4 F'(x + 3h) + \&c.\right\},$$

$$\text{where } N_1 = \frac{1}{12}, \quad N_2 = -\frac{1}{240}, \quad N_3 = \frac{31}{4.5.6.7.8.9}.$$

Put $Fx = -\log \sin x$, $F'x = -\cot x$.

$$\left.\begin{aligned} \therefore \Delta^2 \text{ϟ}x &= -h^2 \{\cot (x + h) \\ &\quad + \tfrac{1}{12}\Delta^2 \cot x - \tfrac{1}{240}\Delta^4 \cot (x - h) + \&c.\} \\ \text{also} \quad &= -h^2 \{\cot (x + h) \\ &\quad + \tfrac{1}{12}\Delta^2 \cot (x + 2h) - \tfrac{1}{240}\Delta^4 \cot (x + 3h) + \&c.\} \end{aligned}\right\} \ldots (31).$$

§. IV.—*Applications of* ϟ.

15. (1) To find $\Theta = -\int_0 \log (\sin^2 \theta - \sin^2 \alpha)\, d\theta,$

$$\text{or} = -\int_0 \log (\cos^2 \alpha - \cos^2 \theta)\, d\theta.$$

Observe that $\sin^2 \theta - \sin^2 \alpha = \sin (\theta + \alpha) . \sin (\theta - \alpha)$.

$$\therefore \quad \Theta = \text{ϟ}(\theta + \alpha) + \text{ϟ}(\theta - \alpha), \quad \text{or} = \text{ϟ}(\alpha + \theta) - \text{ϟ}(\alpha - \theta).$$

(2) To find $\Theta = -\int_0 \log (\cos \theta - \cos \alpha)\, d\theta$.

Since $\cos \theta - \cos \alpha = 2 (\cos^2 \tfrac{1}{2}\theta - \cos^2 \tfrac{1}{2}\alpha)$,

$$\Theta = -\theta \log 2 + 2\text{ϟ}\left(\frac{\alpha + \theta}{2}\right) - 2\text{ϟ}\left(\frac{\alpha - \theta}{2}\right).$$

(3) If $\Theta = \int_0 \log (1 + \sec \mu \cos \theta)\, d\theta,$

since $\log (1 + \sec \mu \cos \theta) = \log \{\cos \mu - \cos (\pi - \theta)\} - \log \cos \mu$,

$$\therefore \Theta = -\theta \log (2 \cos \mu) + 2\text{ϟ}\left(\frac{\mu + \pi - \theta}{2}\right) - 2\text{ϟ}\left(\frac{\mu - \pi + \theta}{2}\right).$$

(4) If $\Theta = \int_0 \log (1 + \sin \mu \cos \theta)\, d\theta$, put $\tan \eta = \dfrac{\sin \theta}{\tan \frac{1}{2}\mu + \cos \theta}$,

and we had by equation (6)

$$\tfrac{1}{2}\Theta = \text{ϟ}(\theta - \eta) + \text{ϟ}\eta - \text{ϟ}\theta + \theta \log \sin \tfrac{1}{2}\mu - \eta \log \tan \tfrac{1}{2}\mu.$$

Thus $\int \log (a \pm b \cos \theta)\, d\theta$ can always be found by ϟ.

(5) If $\Theta = \int_0 \log (\tan \theta + \tan \alpha)\, d\theta$;

$$\text{since } \tan \theta + \tan \alpha = \frac{\sin (\theta + \alpha)}{\cos \theta \cos \alpha},$$

$$\Theta = \text{ϟ}\,\tfrac{1}{2}\pi + \text{ϟ}\alpha - \theta \log \cos \alpha - \text{ϟ}(\theta + \alpha) - \text{ϟ}(\tfrac{1}{2}\pi - \theta).$$

(6) To find $S = 1^{-2}x - 3^{-2}x^3 + 5^{-2}x^5 - 7^{-2}x^7 + \&c.\ldots$ In the process which gave rise to equation (26), put $a = \frac{1}{2}\pi$; $x = \tan\omega$; and observing that $\text{ל}(\omega + \frac{1}{2}\pi) - \text{ל}\frac{1}{2}\pi = \text{ל}\frac{1}{2}\pi - \text{ל}(\frac{1}{2}\pi - \omega)$, we get

$$S = \omega \log x + \text{ל}\omega + \text{ל}(\tfrac{1}{2}\pi - \omega) - \tfrac{1}{2}\text{ל}\pi.$$

The series S received from Spence a special discussion.

(7) If $\Theta = \int \log(1 - 2n\cos a \tan\theta + n^2\tan^2\theta)\, d\theta$, we reduce this to No. (4), as in the process for finding X. See equation (5).

(8) If $\Theta = \int_0 \log(\tan^2\theta + \tan^2\beta)\, d\theta$, make $n = \cot\beta$;

$$\therefore \quad \Theta = 2\theta \log\tan\beta + \int_0 \log(1 + n^2\tan^2\theta)\, d\theta.$$

The last falls under Ex. (7), as a particular case, when $a = \frac{1}{2}\pi$.

(9) If $\Omega = \int_0 \log(1 - 2r\cos a \sin\omega + r^2\sin^2\omega)\, d\omega$, this also may be reduced to ל, by the following process.

Suppose r positive, $x = \tan\frac{1}{2}\omega$, or $\sin\omega = \dfrac{2x}{1 + x^2}$: and $1 + x^2 = \sec^2\frac{1}{2}\omega$;

$$\therefore \Omega = \int_0 \log\{1 - 4rx\cos a + (4r^2 + 2)x^2 - 4rx^3\cos a + x^4\}\, d\omega + 8\text{ל}\left(\frac{\pi - \omega}{2}\right).$$

Assume the quantity under *log* to be

$$= (1 - 2nx\cos\gamma + n^2x^2)(1 - 2n^{-1}x\cos\gamma + n^{-2}x^2);$$

$\therefore (n + n^{-1})\cos\gamma = 2r\cos a$, and $4r^2 + 2 = 4\cos^2\gamma + n^2 + n^{-2}$.

Let $\tan\nu = n$, $\tan\rho = r$; $\therefore 4r^2 + 4 = 4\cos^2\gamma + (n + n^{-1})^2$;

or $\cos\gamma = r\cos a \sin 2\nu$, and $2r\,\text{cosec}\, 2\rho = \cos^2\gamma + \text{cosec}^2\, 2\nu$.

Eliminate γ, and solve for cosec 2ν. The result is, that if we take $\sin 2\zeta = \cos a \sin 2\rho$, and select that root of ζ which makes $\pm\sin\zeta$ least, we have

$$\sin 2\nu = \frac{\cos\rho}{\cos\zeta}, \text{ and } \cos\gamma = \frac{\sin\zeta}{\cos\rho}.$$

Hence, having found ν and γ,

$$\left.\begin{aligned} \text{let } \Omega' &= \int_0 \log(1 - 2n\cos\gamma\tan\tfrac{1}{2}\omega + n^2\tan^2\tfrac{1}{2}\omega)\,\tfrac{1}{2}d\omega \\ \Omega'' &= \int_0 \log(1 - 2n^{-1}\cos\gamma\tan\tfrac{1}{2}\omega + n^{-2}\tan^2\tfrac{1}{2}\omega)\,\tfrac{1}{2}d\omega \end{aligned}\right\}$$

where we pass from Ω' to Ω'' by changing ν to $(\frac{1}{2}\pi - \nu)$; then

Ω becomes $= 2\Omega' + 2\Omega'' + 8\text{ל}\left(\dfrac{\pi - \omega}{2}\right)$.

As in the process of Art. (2), make $\tan\beta = \tan 2\nu \cos\gamma$, $\cos\mu = \sin 2\nu \sin\gamma$, and $\theta = \omega + \beta$; observing that γ and $\frac{1}{2}\omega$ replace a and ω. When ν becomes $(\frac{1}{2}\pi - \nu)$, β becomes $-\beta$ and μ is unchanged. Let $\theta' = \omega - \beta$; then

$$\Omega' = \tfrac{1}{2}\int \log(1 + \sin\mu\cos\theta)\,d\theta - \tfrac{1}{2}\omega\log(2\cos^2\nu) - 2ל\left(\frac{\pi-\omega}{2}\right):$$

whence $\Omega = \int \log(1 + \sin\mu \cos\theta)\,d\theta$

$$+ \int \log(1 + \sin\mu\cos\theta')\,d\theta' - 2\omega\log\sin 2\nu \;..(32),$$

which reduces the case to No. (4).

(10) When F is rational, $\int \log Fx \,.\, \dfrac{dx}{\sqrt{(p^2 - x^2)}}$ is reducible to the sum or difference of integrals such as

$$\int \log(a + bx)\frac{dx}{\sqrt{(p^2 - x^2)}} \text{ and } \int \log(1 - 2x\cos a + x^2)\frac{dx}{\sqrt{(r^2 - x^2)}}.$$

The former falls under No. (3) or No. (4), if $x = p\cos\theta$. The latter is the case of No. (9), if $x = r\cos\omega$. Thus $\int \log Fx \,.\, \dfrac{dx}{\sqrt{(p^2 - x^2)}}$ is wholly reducible to ל.

(11) To find $\displaystyle\int \frac{x^m \log x\, dx}{(a + bx^2)^{n+\frac{1}{2}}}$, when m and n are integers, positive or negative.

Represent the integral by $V_{m,n}$ and let $U_{m,n} = \dfrac{x^m \log x}{(a + bx^2)^{n-\frac{1}{2}}}$. Differentiate $U_{m,n}$ and integrate back again; and we obtain

$$U_{m,n} - \int \frac{x^{m-1}\,dx}{(a + bx^2)^{n-\frac{1}{2}}} = mV_{m-1,n-1} - (2n-1)\,bV_{m+1,n} \;......(1),^*$$

$$= maV_{m-1,n} - (2n - m - 1)\,bV_{m+1,n} \;..(2),^*$$

$$= (2n-1)aV_{m-1,n} - (2n-m-1)bV_{m-1,n-1} ..(3).^*$$

When $m = 0$, the first gives $V_{1,n}$ but fails to reduce $V_{-1,n-1}$ to $V_{1,n}$: also the second then merges in the first. Yet by the third, $V_{-1,n-1}$ can be reduced to $V_{-1,n}$. When $m = 2n - 1$, the second and third coincide, and give $V_{2n-2,n}$, but fail to reduce $V_{2n-2,n-1}$ to $V_{2n-2,n}$.

There being no other cases of failure, and $V_{2n-2,n-1}$ being reducible by the first formula to $V_{\pm4,\pm2}$, $V_{\pm2,\pm1}$ and finally to $V_{0,0}$; it is evident that $V_{m,n}$ is universally reducible to one of the three, $V_{1,0}$, $V_{0,0}$, $V_{-1,0}$, of which the first (being included under $V_{1,n}$) can be found by common methods. This is the case of m being positive and odd. If m is negative and odd, $V_{m,n}$ is reduced to $V_{-1,0}$; if m is even, it is reduced to $V_{0,0}$.

In $V_{-1,0}$ let $x = y^{-1}$; $\therefore \int \frac{\log x\, dx}{x\sqrt{(a+bx^2)}} = \int \frac{\log y\, dy}{\sqrt{(ay^2+b)}}$; which is of the same form as $V_{0,0}$; which alone remains. When $a = 1$ and $b = -1$, put $x = \sin \omega$, $\therefore V_{0,0} = -ㄣ\omega$. When $a = -1$ and $b = 1$, let $y = x + \sqrt{(x^2-1)}$, $\frac{dx}{\sqrt{(x^2-1)}} = \frac{dy}{y}$; $\log x = \log (1+y^2) - \log (2y)$;

$$\therefore \quad V_{0,0} = \int \log (1+y^2)\frac{dy}{y} - \int \log (2y) \frac{dy}{y}$$

$$= \tfrac{1}{2} L\,(1+y^2) - \tfrac{1}{2}\log^2 (2y) + \text{const.}$$

When $\quad a = 1$ and $b = 1$, let $y = x + \sqrt{(x^2+1)}$:

$$\therefore \quad V_{0\,0} = \log y \,.\, \log (y^2-1) - \tfrac{1}{2} L\,(y^2) - \tfrac{1}{2}\log^2 (2y) + \text{const.}$$

To recapitulate then: $V_{m,n}$ is always reducible to ㄣ or L; and in particular, $\int \frac{x^{2m-1}\log x\, dx}{(a+bx^2)^{n+\frac{1}{2}}}$ can be found by circular arcs and logarithms;

$$\int \frac{x^{\pm 2m} \log x\, dx}{(1-x^2)^{n+\frac{1}{2}}} \text{ is reducible to ㄣ}; \quad \int \frac{x^{\pm 2m} \log x\, dx}{(x^2 \pm 1)^{n+\frac{1}{2}}} \text{ to } L;$$

$$\int \frac{x^{-2m+1} \log x\, dx}{(x^2-1)^{n+\frac{1}{2}}} \text{ to ㄣ}; \quad \int \frac{x^{-2m+1} \log x\, dx}{(1 \pm x^2)^{n+\frac{1}{2}}} \text{ to } L;$$

where m and n are integers, m positive and n either positive or negative.

§. V.—*On the Higher Transcendents derivable from* ㄣ.

16. Spence has imagined the integrals L^3, L^4, L^5. . deduced from L^2 or L, by the law $L^n(1+x) = \int L^{n-1}(1+x)\, x^{-1}\, dx$; and has exhibited various fundamental properties of L^n. Put

$$x = e^{2\omega}, \quad \therefore e^{2\omega} + 1 = (e^{\omega} + e^{-\omega})\, e^{\omega},$$

$$\text{or } \log (x+1) = \log (\varepsilon^{\omega} + \varepsilon^{-\omega}) + \omega, \quad \text{and } x^{-1}dx = 2d\omega;$$

$$\therefore \; L(1+x) = \int \log (\varepsilon^{\omega} + \varepsilon^{-\omega})\, 2d\omega + \omega^2.$$

When ω changes to $\omega\sqrt{(-1)}$, the integral here becomes $2\sqrt{(-1)} \,.\, \int \log (2\cos\omega)\, d\omega$, which exhibits the relation which exists between L and ㄣ by imaginaries.

17. Since $d\omega \propto x^{-1}dx$, we may imagine a series of functions $ㄣ^3, ㄣ^4, ㄣ^5$. . analogous to L^3, L^4, L^5. ., by the law $ㄣ^3 x = \int_0 ㄣ x\, dx$, and generally $ㄣ^n x = \int_0 ㄣ^{n-1} x\, dx$; and we now regard ㄣ as virtually $ㄣ^2$. Write also

$$\lambda_2 x = -\int_0 \log x\, dx, \quad \lambda_n x = \int_0 \lambda_{n-1}\, x\, dx;$$

then $\lambda_2 x = x(1 - \log x)$,

$$\lambda_{n+1} x = \frac{x^n}{1.2\ldots n}\left\{1 + \tfrac{1}{2} + \tfrac{1}{3} + \ldots\ldots + \frac{1}{n} - \log x\right\} \ldots (33).$$

Also, as $\text{ל}^2 x = \lambda_2 x + H_1 \dfrac{x^3}{1.3} + H_2 \dfrac{x^5}{2.5} + H_3 . \dfrac{x^7}{3.7} + \&c.\ldots\ldots$

if H_n stands for $\pi^{-2n} S_{2n}$; (see equation 19.)

$$\therefore \text{ל}^n x = \lambda_n x + \frac{2H_1 x^{n+1}}{2.3\ldots(n+1)} + \frac{2H_2 x^{n+3}}{4.5\ldots(n+3)} + \frac{2H_3 x^{n+4}}{6.7\ldots(n+5)} + \&c. \ldots\ldots(34).$$

18. Since $2\text{ל}^2(\tfrac{1}{2}x) = xl2 + 1^{-2}\sin x + 2^{-2}\sin 2x + 3^{-2}\sin 3x + \&c.\ldots$ perpetual integration, with suitable addition of constants, gives

$$\left.\begin{aligned} &2^{2n-1}.\text{ל}^{2n}(\tfrac{1}{2}x) = \frac{x^{2n-1}l2}{1.2\ldots(2n-1)} \\ &+ \frac{x^{2n+3}S_3}{1.2\ldots(2n-3)} - \frac{x^{2n-5}S_5}{1.2\ldots(2n-5)} + \&c.\ldots \pm \frac{x}{1} S_{2n-1} \\ &- 1^{-2n}\sin(x - 2n.\tfrac{1}{2})\pi - 2^{-2n}\sin(2x - 2n\,\tfrac{1}{2}\pi) \\ &\qquad - 3^{-2n}\sin(3x - 2n\,\tfrac{1}{2}\pi) - \&c. \end{aligned}\right\} \ldots (35).$$

$$\left.\begin{aligned} &2^{2n}\text{ל}^{2n+1}(\tfrac{1}{2}x) = \frac{x^{2n}l2}{1.2..2n} \\ &+ \frac{x^{2n-2}S_3}{1.2..(2n-2)} - \frac{x^{2n-4}S_5}{1.2..(2n-4)} + \&c.\ldots \mp S_{2n+1} \\ &- 1^{-2n-1}\sin\{x - (2n+1).\tfrac{1}{2}\pi\} \\ &\qquad - 2^{-2n-1}\sin\{2x - (2n+1)\,\tfrac{1}{2}\pi\} - \&c.\ldots \end{aligned}\right\} \ldots (36).$$

And if in these we put $x = 2\pi$, we get

$$\text{ל}^3\pi = \frac{\pi^2 l2}{1.2} : \quad \text{ל}^4\pi = \frac{\pi^3 l2}{1.2.3} + 2^{-2}.\frac{\pi}{1} S_3 :$$

$$\text{ל}^5\pi = \frac{\pi^4 l2}{1.2.3.4} + 2^{-2}.\frac{\pi^2}{1.2}.S_3 :$$

$$\text{ל}^6\pi = \frac{\pi^5 l2}{1.2..5} + 2^{-2}.\frac{\pi^3}{1.2.3} S_3 - 2^{-4}.\frac{\pi}{1} S_5 :$$

$$\text{ל}^7\pi = \frac{\pi^6 l2}{1.2..6} + 2^{-2}.\frac{\pi^4}{1.2.3.4} S_3 - 2^{-4}.\frac{\pi^2}{1.2}.S_5 :$$

The law is evident. After the two first terms, the signs are alternate. Thus $\text{ל}^n\pi$ is known.

19. If in equations (35), (36) we make $x = \pi$; and with reference to the latter, observe that

$$1^{-m} - 2^{-m} + 3^{-m} - 4^{-m} + \&c.. . = (1 - 2^{-m+1})\, S_m;$$

we obtain

$$2^{2n-1}\mathcal{L}^{2n}(\tfrac{1}{2}\pi) = \frac{\pi^{2n-1}l2}{1.2..(2n-1)}$$
$$+ \frac{\pi^{2n-3}S_3}{1.2..(2n-3)} - \frac{\pi^{2n-5}S_5}{1.2..(2n-5)} + \ldots . \pm \frac{\pi}{1} S_{2n-1} \ldots .(37):$$

$$2^{2n}\mathcal{L}^{2n+1}(\tfrac{1}{2}\pi) = \frac{\pi^{2n}l2}{1.2..2n} + \frac{\pi^{2n-2}S_3}{1.2..(2n-2)} - \&c.. . . .$$
$$\mp (2 - 2^{-2n})\, S_{2n+1} \ldots .(38):$$

by which $\mathcal{L}^n(\tfrac{1}{2}\pi)$ is known.

20. If we perpetually integrate $\mathcal{L}^2 x + \mathcal{L}^2(\pi - x) = \mathcal{L}^2\pi$, we get

$$\mathcal{L}^n x + (-1)^n . \mathcal{L}^n(\pi - x) = \frac{x^{n-2}\mathcal{L}^2\pi}{1.2..(n-2)} - \frac{x^{n-3}\mathcal{L}^3\pi}{1.2..(n-3)} + \ldots .$$
$$+ (-1)^n . \mathcal{L}^n\pi \ldots .(39),$$

which reduces $\mathcal{L}^n x$ to $\mathcal{L}^n(\pi - x)$.

21. Perpetually integrate

$$\tfrac{1}{2}\mathcal{L}^2(2x) = \mathcal{L}^2 x - \mathcal{L}^2(\tfrac{1}{2}\pi - x) + \mathcal{L}^2(\tfrac{1}{2}\pi) - xl2;$$

$$\therefore\ 2^{-n+1}\mathcal{L}^n(2x) = \mathcal{L}^n x + (-1)^{n-1}\mathcal{L}^n(\tfrac{1}{2}\pi - x)$$
$$- \frac{x^{n-1}l2}{1.2..(n-1)} + \frac{x^{n-2}\mathcal{L}^2\tfrac{1}{2}\pi}{1.2..(n-2)} - \frac{x^{n-3}\mathcal{L}^3\tfrac{1}{2}\pi}{1.2..(n-3)} .. \pm \mathcal{L}^n\tfrac{1}{2}\pi ..(40),$$

which may be used exactly as equation (14) in Art. 9.

Make $x = \tfrac{1}{2}\pi$, and multiply by 2^{n-1};

$$\therefore\ \mathcal{L}^n\pi = 2^{n-1}\{1 + (-1)^{n-1}\}\,\mathcal{L}^n\tfrac{1}{2}\pi - \frac{\pi^{n-1}l2}{1.2..(n-1)}$$
$$+ \frac{2\pi^{n-2}\mathcal{L}^2\tfrac{1}{2}\pi}{1.2..(n-2)} - \frac{2^2 . \pi^{n-3}\mathcal{L}^3\tfrac{1}{2}\pi}{1.2..(n-3)} + \ldots . \pm 2^{n-1}\mathcal{L}^n\tfrac{1}{2}\pi \ldots .(41).$$

22. To complete the view of $\mathcal{L}^n x$, we ought to embrace the cases of $x < 0$ and $x > \pi$.

It is obvious that

$$\mathcal{L}^{2n}(-x) = -\mathcal{L}^{2n}x; \text{ but } \mathcal{L}^{2n+1}(-x) = \mathcal{L}^{2n+1}x \ldots . (42).$$

Also, by perpetual integration of $\mathcal{L}^2(n\pi + x) = \mathcal{L}^2(n\pi) + \mathcal{L}^2 x$,

$$\mathcal{L}^m(n\pi + x) = \mathcal{L}^m(n\pi) + \frac{x}{1}\mathcal{L}^{m-1}(n\pi) + \frac{x^2}{1.2}\mathcal{L}^{m-2}(n\pi) + \ldots$$
$$+ \frac{x^{m-2}}{1.2 \ldots (m-2)} . \mathcal{L}^2(m\pi) + \mathcal{L}^m x \ldots . (43).$$

But we farther want to express $\text{ל}^m(n\pi)$ by means of $\text{ל}^m\pi$, $\text{ל}^{m-1}\pi, \ldots \text{ל}^2\pi$, for which we begin with ל^3 and proceed to ל^4, ל^5 in succession.

For $\text{ל}^3(n\pi)$, let $x = \pi$, and $n = 1, 2, 3, 4 \ldots.$

$$\therefore \text{ל}^3(2\pi) = \text{ל}^3(\pi + \pi) = \text{ל}^3\pi + \frac{\pi}{1}\text{ל}^2\pi + \text{ל}^3\pi = 2\text{ל}^3\pi + \frac{\pi}{1}\text{ל}^2\pi,$$

$$\text{ל}^3(3\pi) = \text{ל}^3(2\pi + \pi) = \text{ל}^3(2\pi) + \frac{\pi}{1}\text{ל}^2(2\pi) + \text{ל}^3\pi$$

$$= 3\text{ל}^3\pi + 3\,\frac{\pi}{1}\text{ל}^2\pi,$$

$$\text{ל}^3(4\pi) = \text{ל}^3(3\pi) + \frac{\pi}{1}\text{ל}^2(3\pi) + \text{ל}^3\pi = 4\text{ל}^3\pi + 6\,\frac{\pi}{1}\text{ל}^2\pi.$$

Generally, $$\text{ל}^3(n\pi) = n\text{ל}^3\pi + n\,.\,\frac{n-1}{2}\,.\,\frac{\pi}{1}\text{ל}^2\pi.$$

For $\text{ל}^4(n\pi)$ proceed by the same steps:

$$\text{ל}^4(2\pi) = 2\text{ל}^4\pi + \frac{\pi}{1}\text{ל}^3\pi + \frac{\pi^2}{1.2}\text{ל}^2\pi:$$

$$\text{ל}^4(3\pi) = 3\text{ל}^4\pi + 3\,\frac{\pi}{1}\text{ל}^3\pi + 5\,\frac{\pi^2}{1.2}\text{ל}^2\pi.$$

Generally, $$\text{ל}^4(n\pi) = n\text{ל}^4\pi + \Sigma n\,.\,\frac{\pi}{1}\text{ל}^3\pi + \Sigma n^2\,.\,\frac{\pi^2}{1.2}\,.\,\text{ל}^2\pi.$$

It is easy to see that we may assume

$$\text{ל}^m(n\pi) = n\text{ל}^m\pi + \psi_1 n\,.\,\frac{\pi}{1}\text{ל}^{m-1}\pi + \psi_2 n\,.\,\frac{\pi^2}{1.2}\text{ל}^{m-2}\pi + \ldots$$

to $(m-1)$ terms,

and that the functions ψ do not contain (m). To determine their law, put $x = \pi$ in equation (43), so as to obtain $\text{ל}^m\{(n+1)\,.\,\pi\}$. Also write $(n+1)$ for n in the assumed series, and compare the results. This gives

$$\Delta\psi_1 n = n, \qquad \Delta\psi_2 n = n + 2\psi_1 n,$$

$$\Delta\psi_3 n = n + 3\psi_1 n + 3\psi_2 n, \qquad \Delta\psi_4 n = n + 4\psi_1 n + 6\psi_2 n + 4\psi_3 n,$$

$$\Delta\psi_5 n = n + 5\psi_1 n + 10\psi_2 n + 10\psi_3 n + 5\psi_4 n,$$

where the law is obvious.

Write N_r for $n\,.\,\frac{n-1}{2}\,.\,\frac{n-2}{3}\ldots\frac{n-r+1}{r}$, and integrate the above;

$$\therefore \psi_1 n = \Sigma n = N_2, \qquad \psi_2 n = N_2 + 2N_3,$$

$$\psi_3 n = N_2 + 3N_3 + 3\{N_3 + 2N_4\}$$

$$= N_2 + 6N_3 + 6N_4,$$

$$\psi_4 n = N_2 + 4N_3 + 6\{N_3 + 2N_4\} + 4\{N_3 + 6N_4 + 6N_5\}$$
$$= N_2 + 14N_3 + 36N_4 + 24N_5.$$

Generally, it is easy to satisfy ourselves that

$$\psi_r n = \Delta 0^r \,.\, N_2 + \Delta^2 0^r \,.\, N_3 + \Delta^3 0^r \,.\, N_4 + \ldots\ldots \text{ to } r \text{ terms};$$
$$\therefore\ \Delta\psi_r n = \Delta 0^r \,.\, N_1 + \Delta^2 0^r \,.\, N_2 + \Delta^3 0^r \,.\, N_3 + \ldots\ldots \text{ to } r \text{ terms},$$
$$= n^r. \qquad \text{Hence } \psi_r n = \Sigma n^r.$$

Finally then we obtain

$$\daleth^m (n\pi) = n\daleth^m \pi + \Sigma n \,.\, \frac{\pi}{1} \,\daleth^{m-1}\pi + \Sigma n^2 \,.\, \frac{\pi^2}{1.2} \,.\, \daleth^{m-2}\pi + \ldots\ldots$$
$$\ldots\ldots \text{ to } (m-1) \text{ terms } \ldots\ldots (44),$$

where $\qquad \Sigma n^r = 0^r + 1^r + 2^r + \ldots\ldots (n-1)^r.$

[*To be continued.*]

ON THE LAWS OF EQUILIBRIUM AND MOTION OF SOLID AND FLUID BODIES.

By Samuel Haughton.

(*Continued from* Vol. I. p. 173.)

The differential equations of motion of solid bodies are deduced from (11), by writing $X - \dfrac{d^2\xi}{dt^2}$, $Y - \dfrac{d^2\eta}{dt^2}$, $Z - \dfrac{d^2\zeta}{dt^2}$, for X, Y, Z, and consequently are the following:

$$\epsilon \frac{d^2\xi}{dt^2} = \epsilon X + P, \quad \epsilon \frac{d^2\eta}{dt^2} = \epsilon Y + Q, \quad \epsilon \frac{d^2\zeta}{dt^2} = \epsilon Z + R \ldots (12).$$

Let us suppose that no external forces of any kind act upon the body, and endeavour to satisfy the equations of motion by the particular integral for plane waves,

$$\xi = \cos \alpha \,.\, f(\omega), \quad \eta = \cos \beta \,.\, f(\omega), \quad \zeta = \cos \gamma \,.\, f(\omega),$$
$$\omega = lx + my + nz - vt.$$

Substituting these values of ξ, η, ζ in the differential equations

$$\epsilon \frac{d^2\xi}{dt^2} = P, \quad \epsilon \frac{d^2\eta}{dt^2} = Q, \quad \epsilon \frac{d^2\zeta}{dt^2} = R,$$

we shall obtain the following equations of condition among the constants,

$$\left.\begin{aligned} \epsilon v^2 \,.\, \cos \alpha &= p' \cos \alpha + h' \cos \beta + g' \cos \gamma, \\ \epsilon v^2 \,.\, \cos \beta &= q' \cos \beta + f' \cos \gamma + h' \cos \alpha, \\ \epsilon v^2 \,.\, \cos \gamma &= r' \cos \gamma + g' \cos \alpha + f' \cos \beta, \end{aligned}\right\} \ldots (13),$$

where

$$\left.\begin{aligned}
p' &= Al^2 + Nm^2 + Mn^2 + 2\alpha_1 mn + 2\alpha_2 ln + 2\alpha_3 lm,\\
q' &= Bm^2 + Ln^2 + Nl^2 + 2\beta_1 mn + 2\beta_2 ln + 2\beta_3 lm,\\
r' &= Cn^2 + Ml^2 + Lm^2 + 2\gamma_1 mn + 2\gamma_2 ln + 2\gamma_3 lm,\\
f' &= \alpha_1 l^2 + \beta_1 m^2 + \gamma_1 n^2 + 2L\,mn + 2\gamma_3 ln + 2\beta_2 lm,\\
g' &= \alpha_2 l^2 + \beta_2 m^2 + \gamma_2 n^2 + 2\gamma_3 mn + 2M ln + 2\alpha_1 lm,\\
h' &= \alpha_3 l^2 + \beta_3 m^2 + \gamma_3 n^2 + 2\beta_2 mn + 2\alpha_1 ln + 2N lm,
\end{aligned}\right\} \ldots (14).$$

In the particular integral for plane waves, the direction of the wave (l, m, n) is given; and our object is to determine from the equations of condition among the constants (13), *real* values of $(\alpha, \beta, \gamma, v)$ which denote the direction of the molecular vibration and the velocity of the plane wave. In the present instance this is possible; for it is a well-known property of surfaces of the second degree, that if the equation of the surface be

$$p'x^2 + q'y^2 + r'z^2 + 2f'yz + 2g'xz + 2h'xy = 1 \ldots (15),$$

the equations (13) will determine the directions of the axes of this surface; and, as every surface of the second order has three principal diametral planes, it is evident that there will be three possible directions of molecular vibration, at right angles to each other, for which the direction of the wave plane will be the same, but the velocity of wave propagation will be different; for if (a, b, c) be the axes of the ellipsoid (15), we know (vide *Leroy*, pp. 73, 156) that

$$\epsilon v_{,}^{2} = \frac{1}{a^2}, \quad \epsilon v_{,,}^{2} = \frac{1}{b^2}, \quad \epsilon v_{,,,}^{2} = \frac{1}{c^2},$$

$(v_{,}, v_{,,}, v_{,,,})$ being the three velocities of wave propagation.

Hence we may deduce the following geometrical construction for the directions of molecular vibration corresponding to a given wave plane, and for the velocities of wave propagation.

Construct the six fixed ellipsoids,

$$\left.\begin{aligned}
p &= Ax^2 + Ny^2 + Mz^2 + 2\alpha_1 yz + 2\alpha_2 xz + 2\alpha_3 xy = 1,\\
q &= By^2 + Lz^2 + Nx^2 + 2\beta_1 yz + 2\beta_2 xz + 2\beta_3 xy = 1,\\
r &= Cz^2 + Mx^2 + Ly^2 + 2\gamma_1 yz + 2\gamma_2 xz + 2\gamma_3 xy = 1,\\
f &= \alpha_1 x^2 + \beta_1 y^2 + \gamma_1 z^2 + 2Lyz + 2\gamma_3 xz + 2\beta_2 xy = 1,\\
g &= \alpha_2 x^2 + \beta_2 y^2 + \gamma_2 z^2 + 2\gamma_3 yz + 2Mxz + 2\alpha_1 xy = 1,\\
h &= \alpha_3 x^2 + \beta_3 y^2 + \gamma_3 z^2 + 2\beta_2 yz + 2\alpha_1 xz + 2Nxy = 1,
\end{aligned}\right\} (16),$$

and from their common centre draw a normal to the wave plane; this will pierce the surfaces in six points: let the cor-

responding radii vectores be $(\rho_{\prime}, \rho_{\prime\prime}, \rho_{\prime\prime\prime}, r_{\prime}, r_{\prime\prime}, r_{\prime\prime\prime})$; with these construct the ellipsoid

$$\frac{x^2}{\rho_{\prime}^2} + \frac{y^2}{\rho_{\prime\prime}^2} + \frac{z^2}{\rho_{\prime\prime\prime}^2} + 2\left(\frac{yz}{r_{\prime}^2} + \frac{xz}{r_{\prime\prime}^2} + \frac{xy}{r_{\prime\prime\prime}^2}\right) = 1 \ldots . (17).$$

The directions of the axes of this ellipsoid are the directions of the three possible vibrations of molecules for the given wave, and the three velocities of propagation are inversely proportional to the lengths of the axes.

If wave normals were drawn from the common centre of the ellipsoids (16) in every possible direction, and the corresponding ellipsoids (17) constructed for each direction; and if on each normal three intercepts were measured, inversely proportional to the axes of the corresponding ellipsoid, the extremities of these intercepts would form a surface which would be the *surface of wave velocity*, or locus of feet of perpendiculars from centre on tangent planes of wave surface, and the surface formed by producing the radii vectores of the surface of wave velocity, so that the new radii vectores should be the reciprocals of the old radii, would be the *surface of wave slowness*, or the reciprocal polar of the wave surface; and a knowledge of its properties would serve all the purposes of a knowledge of the wave surface itself. These two surfaces of wave slowness and wave velocity may be determined by the following considerations.

The cubic equation, whose roots are the squares of the reciprocals of the axes of the ellipsoid (15), is

$$(p'-s)(q'-s)(r'-s) - f'^2(p'-s) - g'^2(q'-s) - h'^2(r'-s) + 2f'g'h' = 0.$$

If in this equation we substitute $(x^2 + y^2 + z^2)$ for s, and $\left(\frac{x}{\rho}, \frac{y}{\rho}, \frac{z}{\rho}\right)$ for l, m, n, we shall have the equation of the surface of wave velocity; and if a radius vector be drawn from the centre, it will pierce this surface in three points, and the lengths of the three radii vectores will measure the three velocities of wave propagation possible for the given direction. This surface, however, being only the locus of feet of perpendiculars on tangent planes to wave surface, is of very little use; but it enables us to find the reciprocal polar of the wave surface. Changing the radii vectores into their reciprocals, we obtain

$$(p'\rho^2 - 1)(q'\rho^2 - 1)(r'\rho^2 - 1) - f'^2\rho^4(p'\rho^2 - 1) - g'^2\rho^4(q'\rho^2 - 1) \\ - h'^2\rho^4(r'\rho^2 - 1) + 2f'g'h' \cdot \rho^6 = 0,$$

$$\text{or } (p-1)(q-1)(r-1) - f^2(p-1) - g^2(q-1) - h^2(r-1) + 2fgh = 0 \ldots . \quad (18).$$

This is the surface of wave slowness of elastic solids, and is evidently of the *sixth* degree, from the values of (p, q, r, f, g, h) (16). It has three sheets corresponding to the three velocities of wave propagation, and determines, not merely the laws of propagation of plane waves in a solid, but also enables us to give a construction for the direction of waves reflected or refracted in passing from one solid to another.

With a point in the surface of separation (supposed plane) as centre, construct the two surfaces of wave slowness for both solids; produce the normal to the incident wave to meet its own surface, and from the point in which it pierces it let fall a perpendicular upon the separating plane: this perpendicular will pierce the surface of wave slowness of the second solid in three points; the lines drawn from these points to the centre are the normals of the three refracted waves, and their lengths are inversely proportional to the wave velocities. The directions of the waves reflected back into the first solid may also easily be found by means of this surface; for we have only to produce the perpendicular backwards to meet the surface of wave slowness of the first solid, and the three lines joining the centre with the three points of intersection will be the normals to the three plane waves reflected back into the first solid. These and all other constructions for the direction of reflected and refracted waves, may be easily proved from the properties of the wave surface and the reciprocal properties of the surface of wave slowness, which, for such purposes as these, answers as well as the wave surface itself. The whole theory of wave surfaces in light or in elastic solids, is only a development of Huygens' construction for uniaxal crystals.

It is important to observe, that the directions of the *wave* and *ray* being both given, (by the radius vector and perpendicular on tangent plane of the surface of wave slowness,) determine completely the direction of molecular vibration in any given case. If the direction of the *wave* only be given, the problem is indeterminate; for three parallel tangent planes may be drawn to the wave surface, each tangent plane being accompanied by its own direction of molecular vibration, and the three directions of vibration being at right angles to each other: but if the direction of the *ray* be also given, there will remain nothing indeterminate, for, the di-

rection of the *ray* being the radius vector of the wave surface, knowing the *ray*, we shall know which of the three tangent planes we must select, and consequently which of the three directions of molecular vibration.

Let us now suppose the solid body to have its molecules so arranged, that at each point they are symmetrically placed around three rectangular planes, so that the molecular forces are in every respect similar in each of the eight regions into which the planes divide the body at each point. It is easily seen that, in this case, the coefficients (α_1, β_1, γ_1, α_2, β_2, γ_2, α_3, β_3, γ_3) of the function V are all zero; and consequently the function in this case will be reduced to the following,

$$2V_1 = A\left(\frac{d\xi}{dx}\right)^2 + B\left(\frac{d\eta}{dy}\right)^2 + C\left(\frac{d\zeta}{dz}\right)^2 + Lu^2 + Mv^2 + Nw^2$$
$$+ 2\left(L\frac{d\eta}{dy}\frac{d\zeta}{dz} + M\frac{d\xi}{dx}\frac{d\zeta}{dz} + N\frac{d\xi}{dx}\frac{d\eta}{dy}\right);$$

and the differential equations of motion will become

$$\left.\begin{aligned}
\epsilon\frac{d^2\xi}{dt^2} &= A\frac{d^2\xi}{dx^2} + N\frac{d^2\xi}{dy^2} + M\frac{d^2\xi}{dz^2} + 2\left(N\frac{d^2\eta}{dx\,dy} + M\frac{d^2\zeta}{dx\,dz}\right)\\
\epsilon\frac{d^2\eta}{dt^2} &= B\frac{d^2\eta}{dy^2} + L\frac{d^2\eta}{dz^2} + N\frac{d^2\eta}{dx^2} + 2\left(L\frac{d^2\zeta}{dy\,dz} + N\frac{d^2\xi}{dx\,dy}\right)\\
\epsilon\frac{d^2\zeta}{dt^2} &= C\frac{d^2\zeta}{dz^2} + M\frac{d^2\zeta}{dx^2} + L\frac{d^2\zeta}{dy^2} + 2\left(M\frac{d^2\xi}{dx\,dz} + L\frac{d^2\eta}{dy\,dz}\right)
\end{aligned}\right\} \quad \ldots\ldots (19).$$

The equation (18) will be the equation of the surface of wave slowness, if we give to (p, q, r, f, g, h) the following values,

$$\left.\begin{aligned}
p &= Ax^2 + Ny^2 + Mz^2, & f &= 2Lyz,\\
q &= By^2 + Lz^2 + Nx^2, & g &= 2Mxz,\\
r &= Cz^2 + Mx^2 + Ly^2, & h &= 2Nxy,
\end{aligned}\right\} \ldots . (20),$$

and its traces on the three principal planes will be

$$(r-1)\{(p-1)(q-1) - h^2\} = 0, \quad z = 0,$$
$$(q-1)\{(p-1)(r-1) - g^2\} = 0, \quad y = 0,$$
$$(p-1)\{(q-1)(r-1) - f^2\} = 0, \quad x = 0,$$

from which it appears that the traces of the surface of wave slowness are composed of two distinct curves, one an ellipse and the other a curve of the fourth degree.

If the surface of wave slowness possess nodes in its principal planes, they will appear in the form of multiple points in its traces; and it may be shewn that it does possess real nodes in one of the principal planes.

For, its equation may be put under any one of the three forms

$$\phi_1 + z^2\psi_1 = 0, \quad \phi_2 + y^2\psi_2 = 0, \quad \phi_3 + z^2\psi_3 = 0,$$

where the values of ϕ_1, ϕ_2, ϕ_3 are

$$\phi_1 = (r-1)\{(p-1)(q-1) - h^2\},$$
$$\phi_2 = (q-1)\{(p-1)(r-1) - g^2\},$$
$$\phi_3 = (p-1)\{(q-1)(r-1) - f^2\};$$

but if the equation of any surface be

$$u = \phi + z^2\psi = 0,$$

it can be shewn that, if the surface $\phi = 0$ has any singular points in the plane $z = 0$, these are also singular points in the surface $u = 0$; for the conditions for singular points are

$$\frac{du}{dx} = 0, \quad \frac{du}{dy} = 0, \quad \frac{du}{dz} = 0,$$

which, if $z = 0$ become

$$\frac{d\phi}{dx} = 0, \quad \frac{d\phi}{dy} = 0, \quad \frac{d\phi}{dz} = 0,$$

which are the conditions for singular points in the surface $\phi = 0$. Hence, if any singular points of the surface $\phi = 0$ exist in the plane $z = 0$, these will be also singular points in the surface $u = 0$.

Applying this principle to the three forms of the surface of wave slowness just given, it appears that the intersections of the three surfaces

$$r - 1 = 0, \quad (p-1)(q-1) - h^2 = 0, \quad z = 0,$$

will be singular points of the surface of wave slowness in the plane $z = 0$; and similarly for the other principal planes.

As the traces in the principal planes consist of a curve of the second and another of the fourth degree, there will be in general eight points of intersection, real or imaginary, and therefore the surface of wave slowness should have twenty-four singular points; but as it is only the *real* singular points which produce any effect in the physical problem, we must ascertain the number and position of the real singular points.

I shall first prove that the curve of the fourth degree consists of two ovals, lying one inside the other, and not having

any point in common. The equation of this curve in the plane $z = 0$ is

$$(Ax^2 + Ny^2 - 1)(By^2 + Nx^2 - 1) - 4N^2x^2y^2 = 0,$$

and its polar equation is

$$\left(A\cos^2 a + N\sin^2 a - \frac{1}{\rho^2}\right)\left(B\sin^2 a + N\cos^2 a - \frac{1}{\rho^2}\right) - 4N^2\sin^2 a\cos^2 a = 0,$$

which is a quadratic equation with respect to $\frac{1}{\rho^2}$: the condition necessary for *equal roots* is

$$\{(A + N)\cos^2 a + (B + N)\sin^2 a\}^2$$
$$= 4(A\cos^2 a + N\sin^2 a)(B\sin^2 a + N\cos^2 a) - 16N^2\sin^2 a\cos^2 a.$$

This equation of condition must give a *real* value for a; arranging it with respect to $\tan a$, it becomes

$$(B - N)^2\tan^4 a + \{2(A + N)(B + N) - 4AB + 12N^2\}\tan^2 a + (A - N)^2 = 0;$$

or, assuming $\omega = (A - N)(B - N) - 4N^2$,

$$(B - N)^2\tan^4 a - 2(\omega - 4N^2)\tan^2 a + (A - N)^2 = 0.$$

Now, in order that this equation should give *real* values for $\tan a$, it must give *real* and *positive* values for $\tan^2 a$; but it can be shewn that its roots, if real, are *negative;* and consequently that no real value exists for $\tan a$.

For, solving the equation with respect to $\tan^2 a$, we obtain

$$(B - N)^2 . \tan^2 a = (\omega - 4N^2) \pm \sqrt{\{(\omega - 4N^2)^2 - (\omega + 4N^2)^2\}};$$

or, $$(B - N)^2 . \tan^2 a = (\omega - 4N^2) \pm \sqrt{(-16N^2 . \omega)}.$$

This equation shews that the condition for real values of $\tan^2 a$, is that ω should be either *zero* or *negative*, and that in either case $\tan^2 a$ will be negative, and therefore the two branches of the curve will not have a real point of intersection. The same result is true of the other curves of the fourth degree in the other principal planes.

The curve of the fourth degree, in the plane $z = 0$, consists of two branches, lying one inside the other, each of them cutting at right angles the axes of coordinates, and the semiaxes of one branch are $\frac{1}{\sqrt{A}}, \frac{1}{\sqrt{B}}$; while the semiaxes of the other branch are equal and each $\frac{1}{\sqrt{N}}$; and similarly for the

other coordinate planes. But the equation of the ellipse in the plane $z = 0$, being

$$Ly^2 + Mx^2 - 1 = 0,$$

this curve will cut the oval whose semiaxes are $\frac{1}{\sqrt{N}}$, if N be intermediate in value between L and M; supposing therefore that M is the mean of the three quantities L, M, N; the ellipse in the plane (x, z) will cut the oval whose semiaxes are $\frac{1}{\sqrt{M}}$, and in the other planes it will lie, in one case, completely outside the oval, and in the other completely inside the oval. Hence there are always *at least* four real singular points on the surface of wave slowness. Whether there be more real singular points will depend upon the relative magnitude of A, B, C compared with L, M, N; and if we assume (as seems probable from its being true in homogeneous solids) that A, B, C are greater than L, M, N, then the ovals whose semiaxes are $\frac{1}{\sqrt{A}}, \frac{1}{\sqrt{B}}, \frac{1}{\sqrt{C}}$, will lie completely inside the ellipses and the other ovals whose semiaxes are $\frac{1}{\sqrt{L}}, \frac{1}{\sqrt{M}}, \frac{1}{\sqrt{N}}$. The surface of wave slowness will therefore consist of three sheets; one whose semiaxes are $\frac{1}{\sqrt{A}}, \frac{1}{\sqrt{B}}, \frac{1}{\sqrt{C}}$, isolated, and lying inside the other two sheets; and the other two sheets, having four points in common, like Fresnel's wave surface, and cutting off from the axis of x intercepts equal to $\frac{1}{\sqrt{M}}, \frac{1}{\sqrt{N}}$, from the axis of y, $\frac{1}{\sqrt{L}}, \frac{1}{\sqrt{N}}$; and from the axis of z, $\frac{1}{\sqrt{L}}, \frac{1}{\sqrt{M}}$.

To shew the effect produced by the existence of these nodes in the surface of wave slowness, it is necessary to consider a plane wave in its passage from one solid into another; the construction for the refracted wave is as follows: describe the surfaces of wave slowness (S, Σ) for both solids, having a common centre in the plane which separates the bodies; produce the normal to the incident wave to meet the corresponding sheet of the surface S, and from the point of intersection let fall a perpendicular on the separating plane; this perpendicular will in general pierce the surface Σ in three points; and the corresponding radii vectores will be the normals to the three refracted *waves*, the perpendiculars on the three tangent planes at the points of intersection being the directions of the refracted *rays*. Let us suppose

that the perpendicular pierces the surface Σ in a node, then the line joining the centre with the node will be the normal to the refracted wave; but there will be an infinite number of rays, which will form the sides of a cone of the second degree, having its vertex at the centre of the surface of wave slowness, and reciprocal to the tangent cone at the node. Again, there may be only one refracted ray, and an infinite number of waves; for if we consider that there are four tangent planes of the surface Σ, which touch the surface along an ellipse, it is evident that there might be a cone of refracted wave normals of the second degree, whose base is one of the ellipses of contact, while there would be only one refracted ray, whose direction is the perpendicular to the plane of ellipse: to find, in this case, the cone of incident waves which will be refracted into a single ray, we must project (by perpendiculars to the plane of separation) the ellipse of contact of the surface Σ upon S; then the cone, whose base is the projection and vertex the centre of surface, will be the cone of incident wave normals; while the cone whose base is the ellipse will be the cone of refracted wave normals; and the perpendicular to the plane of the ellipse will be the unique direction of the refracted ray.

In the integration of the differential equations of motion of elastic solids, I have been obliged to use a particular integral which will only represent the case of plane waves; but the differential equations themselves are general, and if a complete integral could be found for them, we should have all the knowledge we could desire upon the subject. It may be observed that I have confined my attention in this paper exclusively to the *laws of propagation*, and such knowledge of *reflection and refraction* as these laws afford: the full investigation of the latter subject is to be sought in the double integrals of which the quantity Δ, in equation (10), is composed. The surface of wave slowness and the laws of wave propagation can only give the *directions* of the reflected and refracted rays and waves, while the laws which regulate the *intensities* of the molecular vibrations, in passing from one solid to another, are to be sought in the conditions at the limits, which are all contained in the quantity Δ. For an attempt to investigate these conditions, and for a fuller account of the laws of propagation, I may refer to a paper read before the Royal Irish Academy, in May 1846, and which will shortly appear in the Transactions of that body.

January 4, 1847.

ON CERTAIN DEFINITE INTEGRALS SUGGESTED BY PROBLEMS IN THE THEORY OF ELECTRICITY.

By WILLIAM THOMSON.

IT follows from the solution of the problem of the distribution of electricity on an infinite plane,* subject to the influence of an electrical point, that the value of the double integral,

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\frac{z\,d\xi\,d\eta}{\{(\xi-x)^2+(\eta-y)^2+z^2\}^{\frac{3}{2}}\{(\xi-x')^2+(\eta-y')^2+z'^2\}^{\frac{1}{2}}},$$

is

$$\frac{2\pi}{\{(x-x')^2+(y-y')^2+(z+z')^2\}^{\frac{1}{2}}}.$$

A direct analytical verification of this result is therefore interesting in connexion with the physical problem. In the following paper the multiple integral

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\frac{u\,d\xi_1\,d\xi_2\,..\,d\xi_n}{\{(\xi_1-x_1)^2+(\xi_2-x_2)^2+..+u^2\}^{\frac{1}{2}(n+1)}\{(\xi_1-x_1')^2+(\xi_2-x_2')^2+..+u'^2\}^{\frac{1}{2}(n-1)}}$$

is considered, and its value is shewn to be

$$\frac{\pi^{\frac{1}{2}(n+1)}}{\Gamma\frac{1}{2}(n+1)}\;\frac{1}{\{(x_1-x_1')^2+(x_2-x_2')^2+\ldots+(u+u')^2\}^{\frac{1}{2}(n-1)}},$$

a result of which the one mentioned above is a particular case. Several distinct demonstrations of this theorem are given, and some other formulæ, which have occurred to me in connexion with it, are added.

The first part of the following paper, which is a translation, with slight alterations, of a memoir in *Liouville's Journal*,† contains a demonstration suggested to me by a method followed by Green in proving the remarkable theorem in Art. (5) of his Essay on Electricity. In the second part some formulæ are given which, in the case of two variables, are such as would occur in the analysis of problems in heat and electricity, with reference to a body bounded in one direction by an infinite plane, if the methods indicated by Fourier were followed; and from them the value of the multiple integral mentioned above is deduced. In §. III. the evaluation is effected by a direct process of reduction, suggested by geometrical considerations.‡

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* See below, end of §. II.

† Vol. x. p. 137, "Démonstration d'un Théorème d'Analyse." April 1845.

‡ See "Extrait d'une lettre à M. Liouville, &c." Vol. x. p. 364.

I.

Let value of the multiple integral, which, if we use a very convenient notation analogous to that of factorials, may be written thus,

$$\left[\int_{-\infty}^{\infty}\right]^n \frac{[d\xi]^n}{\{\Sigma(\xi - x)^2 + u^2\}^{\frac{1}{2}(n+1)}\{\Sigma(\xi - x')^2 + u'^2\}^{\frac{1}{2}(n-1)}},$$

be denoted by U.

Let $u + u' = a$, it being understood that u and u' are taken as positive. Then, if we assume

$$R = \frac{1}{\{\Sigma(\xi - x)^2 + v^2\}^{\frac{1}{2}(s-1)}} - \frac{1}{\{\Sigma(\xi - x)^2 + (2u - v)^2\}^{\frac{1}{2}(s-1)}} \;\ldots (1),$$

$$R' = \frac{1}{\{\Sigma(\xi - x')^2 + (a - v)^2\}^{\frac{1}{2}(s-1)}} \;\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (2),$$

we have $-2(s-1)\,uU = \left[\int_{-\infty}^{\infty}\right]^s R'\dfrac{dR}{dv}\,[d\xi]^s$, when $v = u$.

It is easily seen that the second member of this equation vanishes when $v = \pm\infty$, and that it does not become infinite, even when one of the values 0, $2u$, or a is assigned to u. Hence the preceding equation may be written

$$-2(n-1)\,uU = \int_{-\infty}^{u}\left[\int_{-\infty}^{\infty}\right]^s\left(\frac{dR'}{dv}\,\frac{dR}{dv} + R'\frac{d^2R}{dv^2}\right)[d\xi]^s\,dv.$$

But we have

$$\int\left[\int_{-\infty}^{\infty}\right]^s \frac{dR'}{dv}\,\frac{dR}{dv}\,[d\xi]^s\,dv = \left[\int_{-\infty}^{\infty}\right]^s\int\frac{dR'}{dv}\,\frac{dR}{dv}\,dv\,[d\xi]^s$$

$$= \left[\int_{-\infty}^{\infty}\right]^s R\,\frac{dR'}{dv}\,[d\xi]^s - \int\left[\int_{-\infty}^{\infty}\right]^s R\,\frac{d^2R'}{dv}\,[d\xi]^s\,dv.$$

When we take the integral with respect to v between the limits $-\infty$ and u, the first term vanishes, since at each limit $R = 0$. Thus the preceding equation is reduced to

$$-2(s-1)\,uU = \int_{-\infty}^{u}\left[\int_{-\infty}^{\infty}\right]^s\left(R'\frac{d^2R}{dv^2} - R\,\frac{d^2R'}{dv^2}\right)[d\xi]^s\,dv.$$

Now we have $\qquad \dfrac{d^2R'}{dv^2} + \Sigma\,\dfrac{d^2R'}{d\xi^2} = 0,$

for all values of $\xi_1, \xi_2 \ldots$, provided v be not equal to a. Hence this equation is satisfied for all the values of the

variables between the limits of the integration in the preceding expression, and we may therefore employ it to eliminate $\frac{d^2R'}{dv^2}$: we thus obtain

$$-2(s-1)\,uU = \int_{-\infty}^{u}\left[\int_{-\infty}^{\infty}\right]^{s}\left(R'\frac{d^2R}{dv^2} + R\Sigma\frac{d^2R'}{d\xi^2}\right)[d\xi]^{s}\,dv.$$

Taking one of the terms of the second member, and integrating by parts, we have

$$\int_{-\infty}^{u}\left[\int_{-\infty}^{\infty}\right]^{s} R\frac{d^2R'}{d\xi_{,}^{2}}[d\xi]^{s}\,dv$$

$$= \int_{-\infty}^{u}\left[\int_{-\infty}^{\infty}\right]^{s-1}\left(\int_{-}R\frac{d^2R'}{d\xi_{,}^{2}}\,d\xi_{,}\right)[d\xi]^{s-1}\,dv$$

$$= -\int_{-\infty}^{u}\left[\int_{-\infty}^{\infty}\right]^{-1}\left(\int_{-\infty}^{\infty}\frac{dR'}{d\xi_{,}}\,\frac{dR}{d\xi_{,}}\,d\xi_{,}\right)[d\xi]^{s-1}\,dv$$

$$= \int_{-\infty}^{u}\left[\int_{-\infty}^{\infty}\right]^{s} R'\frac{d^2R}{d\xi_{,}^{2}}[d\xi]^{s}\,dv,$$

since the integrated parts vanish at each limit. By applying a similar process to each term under the sign Σ, we find

$$-2(s-1)\,uU = \int_{-\infty}^{u}\left[\int_{-\infty}^{\infty}\right]^{s} R'\left(\frac{d^2R}{dv^2} + \Sigma\frac{d^2R}{d\xi^2}\right)[d\xi]^{s}\,dv.$$

But, if we denote by Q and Q' the two parts of R, in equation (1), so that $R = Q - Q'$, we have

$$\frac{d^2Q'}{dv^2} + \Sigma\frac{d^2Q'}{d\xi^2} = 0$$

for all values of the variables v, $\xi_{,}$, &c. within the limits of integration; hence there remains

$$-2(n-1)\,uU = \int_{-\infty}^{u}\left[\int_{-\infty}^{\infty}\right]^{s} R'\left(\frac{d^2Q}{dv^2} + \Sigma\frac{d^2Q}{d\xi^2}\right)[d\xi]^{s}\,dv.$$

To determine the value of this expression it may be remarked that the quantity under the integral signs vanishes for all values of the variables which differ sensibly from those expressed by

$$v = 0, \quad \xi_1 = x_1, \quad \xi_2 = x_2, \quad \&c.,$$

and moreover, that if we consider separately the terms of the second member, each is found to be a converging integral: it follows that, if we denote by P the value which R' receives

when the variables have these values assigned, we have

$$-2(s-1)\,uU = P\iiint ..\left(\frac{d^2Q}{d^2v} + \Sigma\,\frac{d^2Q}{d\xi^2}\right) dv\,d\xi_1\,d\xi_2..d\xi_s..(3),$$

where the limits of integration must be such as to include the values 0, x_1, x_2, &c., but are otherwise entirely arbitrary. By considering separately the different terms of this expression, and integrating each with respect to the variable to which it is related, without yet assigning the limits of the integration, we find

$$-2(s-1)\,uU = P\left(\iint ..\frac{dQ}{dv}\,d\xi_1\,d\xi_2..+\iint\frac{dQ}{d\xi_1}\,dv\,d\xi_2 + \&c.\right)..(4).$$

Let us now assume

$$\xi_1 = v_1 + x_1,\ \ \xi_2 = v_2 + x_2,\ \ \&c.,$$

and

$$v^2 + v_1^{\ 2} + .. + v_s^{\ 2} = r^2,$$

from which we have

$$Q = \frac{1}{r^{s-1}},\quad \frac{dQ}{dv} = -\frac{s-1}{r^{s+1}}\,v,\quad \frac{dQ}{d\xi_1} = -\frac{s-1}{r^{s+1}}\,v_1,\ \ \&c.$$

The integrations in equation (3) may be extended to all the values of the variables which satisfy the condition

$$v^2 + v_1^{\ 2} + v_2^{\ 2} + .. + v_s^{\ 2} \overline{\overline{<}}\ a^2;$$

and the limits in (4) will then be such as to include all the values which satisfy the equation

$$v^2 + v_1^{\ 2} + v_2^{\ 2} + .. + v_s^{\ 2} = a^2,\quad \text{or}\quad r^2 = a^2,\ \&c.$$

Hence we have in the successive terms the second member of (4),

$$\frac{dQ}{dv} = -\frac{s-1}{a^{s+1}}\,v,\quad \frac{dQ}{dv_1} = -\frac{s-1}{a^{s+1}}\,v_1,\ \ \&c.$$

If in the integrations we only take the positive values of the variables v, v_1, v_2, &c. which satisfy the limiting condition, we must multiply each integral by 2^{s+1}. Thus we have

$$uU = \frac{2^sP}{a^{s+1}}\left(\int\!\!\int..v\,dv_1\,dv_2..dv_s + \int\!\!\int..v_1\,dv\,dv_2..dv_s + \&c.\right)$$

$$= \frac{2^s(s+1)P}{a^{s+1}}\int\!\!\int..(a^2 - v_1^{\ 2} - v_2^{\ 2} - .. - v_s^{\ 2})^{\frac{1}{2}}\,dv_1\,dv_2..dv_s$$

$$= (s+1)P\int\!\!\int..(1 - l_1 - l_2..- l_s)^{\frac{1}{2}}\,l_1^{-\frac{1}{2}}\,l_2^{-\frac{1}{2}}..l_s^{-\frac{1}{2}}\,dl_1\,dl_2..dl_s;$$

in which last expression the limits include all positive values satisfying the condition

$$l_1 + l_2 + .. + l_s \overline{\overline{<}}\ 1,$$

Hence, by Liouville's theorem,

$$uU = (s+1)\,P\,\frac{\Gamma(\frac{1}{2})^s}{\Gamma\frac{1}{2}(s+1)}\cdot\tfrac{1}{2}s\int_0^1 (1-h)^{\frac{1}{2}}\,h^{\frac{1}{2}(s-1)}dh = \frac{\pi^{\frac{1}{2}(s+1)}}{\Gamma\frac{1}{2}(s+1)}P,$$

which gives the required value of the integral U.

If we denote by U' any integral corresponding to U, in which the system of variables $u, x_1, x_2 \dots$ and $u', x_1', x_2' \dots$ are inverted, we shall have $uU = u'U'$, since P is a function symmetrical with respect to the two systems; and we therefore deduce from the preceding result,

$$\left.\begin{aligned} & u\left[\int_{-\infty}^{\infty}\right]^s \frac{[d\xi]^s}{\{\Sigma(\xi-x)^2+u^2\}^{\frac{1}{2}(s+1)}\{\Sigma(\xi-x')^2+u'^2\}^{\frac{1}{2}(s-1)}} \\ =\; & u'\left[\int_{-\infty}^{\infty}\right]^s \frac{[d\xi]^s}{\{\Sigma(\xi-x')^2+u'^2\}^{\frac{1}{2}(s+1)}\{\Sigma(\xi-x)^2+u^2\}^{\frac{1}{2}(s-1)}} \\ =\; & \frac{\pi^{\frac{1}{2}(s+1)}}{\Gamma\frac{1}{2}(s+1)}\;\frac{1}{\{\Sigma(x-x')^2+(u+u')^2\}^{\frac{1}{2}(s-1)}} \end{aligned}\right\}\dots\dots(5).$$

I shall add another demonstration of this theorem, as an application of some remarkable analysis given by Mr. Green in his memoir "On the determination of the exterior and interior attractions of ellipsoids of variable densities."*

$$\text{Let } V = \left[\int_{-\infty}^{\infty}\right]^s \frac{u\,[d\xi]^s}{\{\Sigma(\xi-x)^2+u^2\}^{\frac{1}{2}(s+1)}\{\Sigma(\xi-x')^2+u'^2\}^{\frac{1}{2}(s-1)}}\dots(6),$$

an integral which may also be expressed thus:

$$-\frac{1}{n-1}\frac{d}{du}\left\{\left[\int_{-\infty}^{\infty}\right]^s \frac{[d\xi]^s}{\{\Sigma(\xi-x)^2+u^2\}^{\frac{1}{2}(s-1)}\{\Sigma(\xi-x')^2+u'^2\}^{\frac{1}{2}(s-1)}}\right\}.$$

From this latter form, we see that the equation

$$\frac{d^2V}{du^2} + \Sigma\frac{d^2V}{dx^2} = 0\dots\dots\dots\dots\dots(7)$$

is satisfied, provided u does not vanish. Hence V is a function which satisfies this equation for all values of $x_1, x_2 \dots$ and for all the values of u between 0 and ∞. At these limits the value of V may be easily determined, and the general value inferred in the following manner.

When $u = 0$ the quantity under the signs of integration in the expression for V vanishes for all the values of $\xi_1, \xi_2 \dots$ which are not equal to $x_1, x_2 \dots$ respectively. Hence it follows that, when $u = 0$,

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* Read at the Cambridge Philosophical Society, May 6, 1846. See *Trans.*, vol. v.

$$V = \frac{1}{\{\Sigma(x-x')^2+u'^2\}^{\frac{1}{2}(s-1)}} \left[\int_{-\infty}^{\infty}\right]^s \frac{u\,[d\xi]^s}{\{\Sigma(\xi-x)^2+u^2\}^{\frac{1}{2}(s+1)}}$$

$$= \frac{1}{\{\Sigma(x-x')^2+u'^2\}^{\frac{1}{2}(s-1)}} \left[\int_{-\infty}^{\infty}\right]^s \frac{dz_1\,dz_2\,..dz_s}{(1+z_1^2+z_2^2+..+z_s^2)^{\frac{1}{2}(s+1)}}$$

$$= \frac{1}{\{\Sigma(x-x')^2+u'^2\}^{\frac{1}{2}(s-1)}} \cdot \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \cdot \cdot \frac{l_1^{-\frac{1}{2}}\,l_2^{-\frac{1}{2}}\,..dl_1\,dl_2\,..}{(1+l_1+l_2+..+l_s)^{\frac{1}{2}(s+1)}}$$

$$= \frac{1}{\{\Sigma(x-x')^2+u'^2\}^{\frac{1}{2}(s-1)}} \frac{\pi^{\frac{1}{2}s}}{\Gamma(1+\frac{1}{2}s)} \int_0^{\infty} \frac{\frac{1}{2}s\,h^{\frac{1}{2}(s-1)}}{(1+h)^{\frac{1}{2}(s+1)}}$$

$$= \frac{\pi^{\frac{1}{2}(s+1)}}{\Gamma\frac{1}{2}(s+1)} \frac{1}{\{\Sigma(x-x')^2+u'^2\}^{\frac{1}{2}(s-1)}}.$$

Also, when $u=\infty$, the value of V is nothing.

Thus we see that V has the same value as the expression

$$\frac{\pi^{\frac{1}{2}(s+1)}}{\Gamma\frac{1}{2}(s+1)} \cdot \frac{1}{\{\Sigma(x-x')^2+(u+u')^2\}^{\frac{1}{2}(s-1)}},$$

when $u=0$, and when $u=\infty$; which enables us to infer that

$$V = \frac{\pi^{\frac{1}{2}(s+1)}}{\Gamma\frac{1}{2}(s+1)} \cdot \frac{1}{\{\Sigma(x-x')^2+(u+u')^2\}^{\frac{1}{2}(s-1)}},$$

for all positive values of u, provided u' be taken as positive: for the second member of this equation satisfies equation (7) for all positive values of u, and for any values of the other variables, and at the limits $u=0$, and $u=\infty$ has the same value as V, and therefore, by a theorem of Green's, in the memoir referred to, must be equal to V for all positive values of u.

From what has been proved above we may deduce the solution of the following problem:

Having given for all values of ξ_1, ξ_2 ...., the value of the multiple integral

$$\mathbf{S}\frac{\rho'\,dx_1'\,dx_2'\,..dx_s'}{\{(x_1'-\xi_1)^2+(x_2'-\xi_2)^2+..+(x_s'-\xi_s)^2+u'^2\}^{\frac{1}{2}(s-1)}} \;....(a),$$

where u' and ρ' are any functions of x_1', x_2'....x_s', let it be required to find the value of

$$\mathbf{S}\frac{\rho'\,dx_1'\,dx_2'\,..dx_s'}{\{(x_1'-x_1)^2+(x_2'-x_2)^2+...+(x_s'-x_s)^2+(u'+u)^2\}^{\frac{1}{2}(s-1)}} \;..(b),$$

where x_1, x_2.., x_s are any given quantities, and u a given positive quantity.

Denoting the expression (a) by Φ, and the expression (b) by ϕ, we have, from the theorem established above,

$$\phi = \frac{u\Gamma\frac{1}{2}(s+1)}{\pi^{\frac{1}{2}(s+1)}} \mathbf{S}\, \rho'\, dx_1'\, dx_2' \ldots dx_s' .$$

$$\left[\int_{-\infty}^{\infty}\right]^s \frac{[d\xi]^s}{\{\Sigma(\xi - x)^2 + u^2\}^{\frac{1}{2}(s+1)}\{\Sigma(\xi - x')^2 + u'^2\}^{\frac{1}{2}(s-1)}}$$

$$= \frac{u\Gamma\frac{1}{2}(s+1)}{\pi^{\frac{1}{2}(s+1)}}\left[\int_{-\infty}^{\infty}\right]^s \frac{[d\xi]^s}{\{\Sigma(\xi-x)^2+u^2\}^{\frac{1}{2}(s+1)}} \cdot \mathbf{S} \frac{\rho'\, dx_1'\, dx_2' \ldots dx_s}{\{\Sigma(\xi-x')^2+u'^2\}^{\frac{1}{2}(s-1)}},$$

or $$\phi = \frac{u\Gamma\frac{1}{2}(s+1)}{\pi^{\frac{1}{2}(s+1)}}\left[\int_{-\infty}^{\infty}\right]^s \frac{\Phi\,[d\xi]^s}{\{\Sigma(\xi - x)^2 + u^2\}^{\frac{1}{2}(s+1)}} \ldots . (c).$$

But, by hypothesis, ϕ' is given for all values of $\xi_1, \xi_2 \ldots \xi_s$; and therefore this equation expresses the solution of the problem. We may also deduce from the theorem (5) the expression

$$\phi = -\frac{\Gamma(\frac{1}{2}s+1)}{(s-1)\,\pi^{\frac{1}{2}(s+1)}}\left[\int_{-\infty}^{\infty}\right]^s \frac{\Psi\,[d\xi]^s}{\{\Sigma(\xi - x)^2 + u^2\}^{\frac{1}{2}(s-1)}} \ldots (d),$$

by means of which ϕ may be determined when the value, Ψ, of $\frac{d\phi}{du}$ corresponding to $u = 0$ is given.

For the particular case of $u' = 0$, the theorem (d) is included in a theorem given by Green, in which the number n in the exponent of the denominator may differ from the number s of variables, the sole condition being that $n - s + 1$ must be positive; but it is only in the case of $n = s$ that a general theorem such as (d), by means of which the general value of ϕ is obtained from the value $\frac{d\phi}{du}$ when $u = 0$, is obtained, can be established.

Let us now apply these formulæ to the case of $x = 2$: we may in this case conveniently replace x_1, x_2, u by x, y, z, and ξ_1, ξ_2, by ξ, η. Equations (c) and (d) become

$$\phi = \frac{z}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \frac{\Phi\, d\xi\, d\eta}{\{(\xi - x)^2 + (\eta - \gamma)^2 + z^2\}^{\frac{3}{2}}} \ldots\ldots (e),$$

and $$\phi = -\frac{1}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \frac{\Psi\, d\xi\, d\eta}{\{(\xi - x)^2 + (\eta - \gamma)^2 + z^2\}^{\frac{1}{2}}} \ldots (f),$$

where Ψ denotes the value of $\frac{d\phi}{dz}$ when $x = \xi$, $y = \eta$, $z = 0$.

The first of these theorems may be deduced from a very general theorem given by Green in his essay on Electricity and Magnetism. The second may be demonstrated in the following manner.

Let x', y', z' be considered as the coordinates of a point P', where there is situated a quantity of matter $\rho' \, dx' \, dy' \, dz'$, in the volume $dx' \, dy' \, dz'$. Then ϕ will be the potential on a point P (x, y, z), above the plane of x, y which we may regard as horizontal, due to a quantity of matter,

$$M, (= \iiint \rho' \, dx' \, dy' \, dz')$$

situated below this plane. Now it follows from a theorem, first, so far as I am aware, given by Gauss, for a surface of any form, that there is a determinate distribution of matter upon the plane (xy) which will produce this same potential on points above the plane. Let k be the density of this distribution at a point Π (ξ, η) of the plane, so that

$$\phi = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \frac{k d\xi \, d\eta}{\{(\xi - x)^2 + (\eta - y)^2 + z^2\}^{\frac{1}{2}}},$$

which gives

$$\frac{d\phi}{dz} = -z \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \frac{k \, d\xi \, d\eta}{\{(\xi - x)^2 + (\eta - y)^2 + z^2\}^{\frac{3}{2}}}.$$

Let $z = 0$; then, denoting by k and $\left(\frac{d\phi}{dz}\right)_0$ the values of k and $\frac{d\phi}{dz}$ at the point $(x, y, 0)$, we find

$$\left(\frac{d\phi}{dz}\right)_0 = -\mathrm{k} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \frac{z \, d\xi \, d\eta}{\{(\xi - x)^2 + (\eta - y)^2 + z^2\}^{\frac{3}{2}}},$$
$$= -\mathrm{k} \, . \, 2\pi,$$

since the value of the integral in the second member is 2π, whatever be the value of z. Hence we conclude that $k = -\frac{1}{2\pi} \, . \, \Phi$, and equation (f) is established.

It should be remarked that the total quantity of matter distributed over the plane xy must be equal to the mass M, which it represents: this is readily verified from the preceding formulæ.

The same formulæ admit of an interesting application in the theory of heat. Thus let ϕ be the permanent temperature of a point P in an infinite homogeneous solid, heated by constant sources distributed below the plane (xy), (the case in which some of the sources are in this plane being of course included). If the temperature Φ at any point Π in the plane (xy) be given, the formula (e) enables us to find the temperature at any point above the plane.

As an example, let us suppose that the sources of heat are such that the temperature of a portion A of the plane (xy),

between two lines parallel to OY and at equal distances, a, on its two sides, has a constant value c, and the temperature of the remainder of the plane zero. In this case the formula (e) will give, for the temperature at a point (x, y, z) above the plane,

$$\phi = \frac{zc}{2\pi} \int_{-\infty}^{\infty} \int_{-a}^{a} \frac{d\xi \, d\eta}{\{(\xi - x)^2 + (\eta - y)^2 + z^2\}^{\frac{3}{2}}}$$

$$= \frac{c}{\pi} \left(\tan^{-1} \frac{x + a}{z} - \tan^{-1} \frac{x - a}{z} \right)$$

$$= \frac{c}{\pi} \tan^{-1} \frac{2az}{x^2 + z^2 - a^2}.$$

From this we conclude that the isothermal surfaces which correspond to this case are circular cylinders, which intersect the plane (xy) in the two parallel lines bounding A.

The application to this example, and all others in which the isothermal surfaces are cylindrical, may be made directly by putting $s = 1$ in the general formulæ.

II.

I now proceed to find the values, which will be denoted by V and W, of the integrals

$$\left[\int_{-\infty}^{\infty}\right]^s \frac{[d\xi]^s \, [\cos m\xi]^s}{(\Sigma\xi^2 + u^2)^{\frac{1}{2}(s-1)}}$$

and
$$\left[\int_{-\infty}^{\infty}\right]^s [dm]^s \, [\cos mx]^s \, \frac{\epsilon^{-(\Sigma m^2)^{\frac{1}{2}} u}}{(\Sigma m^2)^{\frac{1}{2}}},$$

where the symbols $[\cos m\xi]^s$, $[\cos mx]^s$ denote the products

$$\cos m_1\xi_1 \, . \cos m_2\xi_2 \, . \, . \cos m_s\xi_s,$$

$$\cos m_1x_1 \, . \cos m_2x_2 \, . \, . \cos m_sx_s;$$

and the notation is in other respects the same as before.

By means of the formula

$$[\cos m\xi + \sin m\xi \, . \sqrt{(-1)}]^s = \cos (\Sigma m\xi) + \sin (\Sigma m\xi) \, . \sqrt{(-1)},$$

it is easily shewn that

$$V = \left[\int_{-\infty}^{\infty}\right]^s \frac{[d\xi]^s \cos \Sigma (m\xi)}{(\Sigma\xi^2 + u^2)^{\frac{1}{2}(s-1)}} \; \ldots\ldots\ldots\ldots \; (a).$$

Hence, by a suitable linear transformation, in which one of the assumptions is $\Sigma m\xi = \eta (\Sigma m^2)^{\frac{1}{2}}$, we have

$$V = \int_{-\infty}^{\infty} \cos \mu\eta \, . \, d\eta \, . \left[\int_{-\infty}^{\infty}\right]^{s-1} \frac{[d\zeta]^{s-1}}{(u^2 + \eta^2 + \Sigma\zeta^2)^{\frac{1}{2}(s-1)}} \; \ldots . (b).$$

Now, by means of Liouville's theorem,* we find

$$\left[\int_{-\infty}^{\infty}\right]^{s-1} \frac{[d\zeta]^{s-1}}{(u^2+\eta^2+\Sigma\zeta^2)^{\frac{1}{2}s(-1)}} = \frac{\pi^{\frac{1}{2}(s-1)}}{\Gamma\,\frac{1}{2}(s-1)} \int_0^{\infty} \frac{2\xi\, d\xi\,.\,\xi^{s-3}}{(\xi^2+\eta^2+u^2)^{\frac{1}{2}(s-1)}}.$$

Hence $$V = \frac{4\pi^{\frac{1}{2}(s-1)}}{\Gamma\,\frac{1}{2}(s-1)} \int_0^{\infty}\!\!\int_0^{\infty} \frac{\xi^{s-2}\cos\mu\eta\,.\,d\xi\,d\eta}{(\xi^2+\eta^2+u^2)^{\frac{1}{2}(s-1)}} \ldots\ldots (c).$$

Differentiating with respect to u, by which the farther reduction of the integral will be facilitated, we have

$$-\frac{dV}{du} = (s-1)\,\frac{4\pi^{\frac{1}{2}(s-1)}}{\Gamma\,\frac{1}{2}(s-1)} \int_0^{\infty}\!\!\int_0^{\infty} \frac{u\,.\,\xi^{s-2}\cos\mu\eta\,.\,d\xi\,d\eta}{(\xi^2+\eta^2+u^2)^{\frac{1}{2}(s+1)}} \,..(d).$$

Now

$$\int_0^{\infty} \frac{\xi^{s-2}\,d\xi}{(\xi^2+\eta^2+u^2)^{\frac{1}{2}(s+1)}} = \int_0^{\infty} \frac{\xi^{-3}\,d\xi}{\left(1+\dfrac{\eta^2+u^2}{\xi^2}\right)^{\frac{1}{2}(s+1)}} = \frac{1}{2}\int_0^{\infty} \frac{dt}{\{1+(\eta^2+u^2)t\}^{\frac{1}{2}(s+1)}}$$

$$= \frac{1}{s-1}\,.\,\frac{1}{\eta^2+u^2}.$$

Hence $$-\frac{dV}{du} = \frac{4\pi^{\frac{1}{2}(s-1)}}{\Gamma\,\frac{1}{2}(s-1)} \int_0^{\infty} \frac{u\cos\mu\eta\,d\eta}{\eta^2+u^2} \ldots\ldots\ldots (e),$$

$$= \frac{2\pi^{\frac{1}{2}(s+1)}}{\Gamma\,\frac{1}{2}(s-1)}\,.\,\epsilon^{-\mu u}.$$

From this, by integration with respect to u, we deduce the value of V: thus we have the result

$$\left[\int_{-\infty}^{\infty}\right]^{s} \frac{[d\xi]^s\,[\cos m\xi]^s}{(\Sigma\xi^2+u^2)^{\frac{1}{2}(s-1)}} = \frac{2\pi^{\frac{1}{2}(s+1)}}{\Gamma\,\frac{1}{2}(s-1)}\,\frac{\varepsilon^{-(\Sigma m^2)^{\frac{1}{2}}u}}{(\Sigma m^2)^{\frac{1}{2}}} \ldots . (V).$$

To evaluate the integral W we may in the first place reduce it to a double integral by a process similar to that indicated above, for obtaining the expression (c); and we thus find

$$W = \frac{4\pi^{\frac{1}{2}(s-1)}}{\Gamma\,\frac{1}{2}(s-1)} \int_0^{\infty}\!\!\int_0^{\infty} \frac{dm\,dn\,.\,m^{s-2}\cos(nr)\,.\,\epsilon^{-(m^2+n^2)^{\frac{1}{2}}u}}{(m^2+n^2)^{\frac{1}{2}}} \,..(a),$$

where r denotes $(\Sigma x^2)^{\frac{1}{2}}$. If we take $m = \rho\cos\vartheta$, $n = \rho\sin\vartheta$, this becomes

$$W = \frac{4\pi^{\frac{1}{2}(s-1)}}{\Gamma\,\frac{1}{2}(s-1)} \int_0^{\infty}\!\!\int_0^{\frac{1}{2}\pi} d\theta\,d\rho\,\rho^{s-2}\cos^{s-2}\vartheta\,\cos(r\rho\sin\vartheta)\,\varepsilon^{-\rho u} \,..(b).$$

Now we have

$$\left(\frac{d^2}{du^2}+\frac{d^2}{dr^2}\right)^{f}\cos(r\rho\sin\vartheta)\,\varepsilon^{-\rho u} = \rho^{2f}\cos^{2f}\vartheta\,.\cos(r\rho\cos\vartheta)\varepsilon^{-\rho u} \,..(c).$$

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* See vol. II. p. 221, First Series.

Considering first the case where s is even, let $f = \frac{1}{2}s - 1$; we thus find

$$\rho^{s-2}\cos^{s-2}\vartheta\cos(r\rho\cos\vartheta)\,\varepsilon^{-\rho u} = \left(\frac{d^2}{du^2}+\frac{d^2}{dr^2}\right)^{\frac{1}{2}s-1}\cos(r\rho\sin\vartheta)\,\varepsilon^{-\rho u},$$

and, by substitution in (b), we have

$$W = \frac{4\pi^{\frac{1}{2}(s-1)}}{\Gamma\frac{1}{2}(s-1)}\int_0^\infty\int_0^{\frac{1}{2}\pi} d\vartheta\, d\rho\,.\left(\frac{d^2}{du^2}+\frac{d^2}{dr^2}\right)^{\frac{1}{2}s-1}\cos(r\rho\sin\vartheta)\,\varepsilon^{-\rho u}$$

$$= \frac{4\pi^{\frac{1}{2}(s-1)}}{\Gamma\frac{1}{2}(s-1)}\left(\frac{d^2}{du^2}+\frac{d^2}{dr^2}\right)^{\frac{1}{2}s-1}\int_0^\infty\int_0^{\frac{1}{2}\pi} d\vartheta\, d\rho\cos(r\rho\sin\vartheta)\,\varepsilon^{-\rho u}$$

$$= \frac{4\pi^{\frac{1}{2}(s-1)}}{\Gamma\frac{1}{2}(s-1)}\left(\frac{d^2}{du^2}+\frac{d^2}{dr^2}\right)^{\frac{1}{2}s-1}\int_0^{\frac{1}{2}\pi}\frac{u\,d\vartheta}{u^2+r^2\sin^2\vartheta}$$

$$= \frac{4\pi^{\frac{1}{2}(s-1)}}{\Gamma\frac{1}{2}(s-1)}\left(\frac{d^2}{du^2}+\frac{d^2}{dr^2}\right)^{\frac{1}{2}s-1}\frac{\frac{1}{2}\pi}{(u^2+r^2)^{\frac{1}{2}}}$$

$$= \frac{4\pi^{\frac{1}{2}(s-1)}}{\Gamma\frac{1}{2}(s-1)}\,\frac{1.3^2\ldots(s-1)^2}{(u^2+r^2)^{\frac{1}{2}(s-1)}} = 2^{s-1}\pi^{\frac{1}{2}(s-1)}\,\Gamma\tfrac{1}{2}(s-1)\,\frac{1}{(u^2+r^2)^{\frac{1}{2}(s-1)}} \quad\ldots\ldots(d).$$

In the second case, when s is odd, let $f = \frac{1}{2}(s-1)$ in (c); then, making use of the result in (b), we have

$$W = \frac{4\pi^{\frac{1}{2}(s-1)}}{\Gamma\frac{1}{2}(s-1)}\left(\frac{d^2}{du^2}+\frac{d^2}{dr^2}\right)^{\frac{1}{2}(s-1)}$$

$$\int_0^\infty\int_0^{\frac{1}{2}\pi} d\vartheta\, d\rho\,\rho\cos\vartheta\,.\cos(r\rho\sin\vartheta)\,\varepsilon^{-\rho u}$$

$$= \frac{4\pi^{\frac{1}{2}(s-1)}}{\Gamma\frac{1}{2}(s-1)}\left(\frac{d^2}{du^2}+\frac{d^2}{dr^2}\right)^{\frac{1}{2}(s-1)}\int_0^\infty d\rho\,.\,\frac{\sin(r\rho)}{r}\,\varepsilon^{-\rho u}$$

$$= \frac{4\pi^{\frac{1}{2}(s-1)}}{\Gamma\frac{1}{2}(s-1)}\left(\frac{d^2}{du^2}+\frac{d^2}{dr^2}\right)^{\frac{1}{2}(s-1)}\frac{1}{r^2+\rho^2}$$

$$= 2^{s-1}\pi^{\frac{1}{2}(s-1)}\,\Gamma\tfrac{1}{2}(s-1)\,\frac{1}{(u^2+r^2)^{\frac{1}{2}(s-1)}}.$$

Hence, whether s be odd or even, we conclude that

$$\left[\int_{-\infty}^{\infty}\right]^s [dm]^s\,[\cos mx]^s\,\frac{\epsilon^{-(\Sigma m^2)^{\frac{1}{2}}u}}{(\Sigma m^2)^{\frac{1}{2}}}$$

$$= 2^{s-1}\pi^{\frac{1}{2}(s-1)}\,\Gamma\tfrac{1}{2}(s-1)\,\frac{1}{(u^2+\Sigma x^2)^{\frac{1}{2}(s-1)}}\quad\ldots.(W).$$

The investigation which we have just gone through, of the integrals (V), (W) constitutes the verification of "Fourier's theorem" in a particular case. For, by this theorem, we

have, if $F(x_1, x_2 . .)$ be a function which remains the same when the signs of any of the variables are changed,

$$2^s \pi^s F(x_1, x_2 . .)$$

$$= \left[\int_{-\infty}^{\infty}\right]^s [dm]^s [\cos mx]^s \int_{-\infty}^{\infty} [d\xi]^s [\cos m\xi]^s F(\xi_1, \xi_2 . .) \;..\; (e);$$

and if we take

$$F(\xi_1, \xi_2 . .) = \frac{1}{(\Sigma\xi^2 + u^2)^{\frac{1}{2}(s-1)}},$$

the result of the integrations with respect to $\xi_1, \xi_2 . .$, is given by (V), and the second member thus becomes a multiple integral with respect to $m_1, m_2 . .$, which is shewn by (W) to be equal to the first member. Conversely, if we assume Fourier's theorem, we may deduce the value W, by means of it, from that of V. The integrals V and W are also connected by means of another case of Fourier's theorem, found by taking, in (e),

$$F(\xi_1, \xi_2 . .) = \frac{\varepsilon^{-(\Sigma\xi^2)^{\frac{1}{2}}u}}{(\Sigma\xi)^{\frac{1}{2}}}.$$

In this way, after the value of W has been found, that of V may be deduced.

The formulæ (V) and (W) may be applied to evaluate the multiple integral u, and we shall thus obtain the result of the investigation in §. I. in a different manner.

By means of the equation obtained by differentiating (W) with respect to u, we find

$$\frac{u}{\{\Sigma(\xi - x)^2 + u^2\}^{\frac{1}{2}(s+1)}} = \frac{1}{2^{s-1}(s-1)\pi^{\frac{1}{2}(s-1)}\Gamma\frac{1}{2}(s-1)}$$

$$\left[\int_{-\infty}^{\infty}\right]^s [dm]^s [\cos m(\xi - x)]^s \varepsilon^{-(\Sigma m^2)^{\frac{1}{2}}u};$$

Making this substitution, for one of the factors of the expression under the integral signs in U, we have

$$Uu = \frac{1}{2^{s-1}(s-1)\pi^{\frac{1}{2}(s-1)}\Gamma\frac{1}{2}(s-1)} \left[\int_{-\infty}^{\infty}\right]^s \frac{[d\xi]^s}{[\Sigma(\xi - x')^2 + u'^2\}^{\frac{1}{2}(s-1)}}$$

$$\left[\int_{-\infty}^{\infty}\right] [dm]^s [\cos m(\xi - x)]^s \varepsilon^{-(\Sigma m^2)^{\frac{1}{2}}u}$$

$$= \frac{1}{2^{s-1}(s-1)\pi^{\frac{1}{2}(s-1)}\Gamma(\frac{1}{2}s - 1)}$$

$$\left[\int_{-\infty}^{\infty}\right]^s [dm]^s [\cos m(x - x')^s]^s \varepsilon^{-(\Sigma m^2)^{\frac{1}{2}}u} \left[\int_{-\infty}^{\infty}\right]^s \frac{[d\xi]^s [\cos m(\xi - x')]^s}{\{\Sigma(\xi - x')^2 + u'^2\}^{\frac{1}{2}(s-1)}}$$

$$= \frac{\pi}{2^{s-2}(s-1)\left\{\Gamma\frac{1}{2}(s-1)\right\}^2}$$

$$\left[\int_{-\infty}^{\infty}\right]^s [dm]^s\,[\cos m(x-x')]^s\,\varepsilon^{-(\Sigma m^2)^{\frac{1}{2}}u}\,\frac{\varepsilon^{-(\Sigma m_2)^{\frac{1}{2}}}u'}{(\Sigma m^2)^{\frac{1}{2}}}, \text{ by } (V),$$

$$= \frac{2\pi^{\frac{1}{2}(s+1)}}{(s-1)\,\Gamma\frac{1}{2}(s-1)}\,\frac{1}{\left\{\Sigma(x-x')^2+(u+u')^2\right\}^{\frac{1}{2}(s-1)}}, \text{ by } (W),$$

which agrees with the value obtained above.

III.

The value of the integral U may also be obtained by a direct process of reduction, as follows.

By a suitable linear transformation, in which assumptions such as

$$\xi_1 - x_1 = \Sigma a\zeta$$

are made, we find

$$U = \left[\int_{-\infty}^{\infty}\right] \frac{[d\zeta]^s}{(\Sigma\zeta^2+u^2)^{\frac{1}{2}(s+1)}\,(\Sigma\zeta^2-2f\zeta_1+f^2+u'^2)^{\frac{1}{2}(s-1)}} \ldots (a),$$

where

$$f^2 = \Sigma(x-x')^2.$$

Let us now assume

$$\zeta_1 = \rho\cos\phi,\quad \zeta_2 = \rho\sin\phi\cos\theta_1,\quad \zeta_3 = \rho\sin\phi\sin\theta_1\cos\theta_2 \ldots,$$
$$\zeta_{s-1} = \rho\sin\phi\sin\theta_1\sin\theta_2\ldots\cos\theta_{s-2},$$
$$\zeta_s = \rho\sin\phi\sin\theta_1\sin\theta_2\ldots\sin\theta_{s-2},$$

from which we deduce*

$$[d\xi]^s = \rho^{s-1}\sin^{s-2}\phi\,\sin^{s-3}\theta_1\,\sin^{s-4}\theta_2\ldots\sin\theta_{s-3}[d\theta]^{s-2}\,d\phi\,d\vartheta;$$

a transformation given first by Green. Equation (a) is thus reduced to

$$U = H_{s-2}\int_0^{\infty}\int_0^{2\pi} \frac{\rho^{s-1}\sin^{s-2}\phi\,d\phi\,d\rho}{(\rho^2+u^2)^{\frac{1}{2}(s+1)}(\rho^2-2\rho f\cos\phi+f^2+u'^2)^{\frac{1}{2}(s-1)}} \ldots (b),$$

where H_{s-2} denotes the product

$$\int_0^{\pi}\sin^{s-3}\theta\,d\theta\,.\int_0^{\pi}\sin^{s-4}\theta\,d\theta\ldots\int_0^{\pi}d\theta.$$

Let $\rho = u\tan\frac{1}{2}\vartheta$; we thus get

$$Uu = \tfrac{1}{2}H_{s-2}\int_0^{\pi}\int_0^{2\pi}$$

$$\frac{\sin^{s-1}\vartheta\,\sin^{s-2}\phi\,d\phi\,d\vartheta}{\left\{2(f^2+u'^2+u^2)+2(f^2+u'^2-u^2)\cos\vartheta-4uf\sin\vartheta\cos\phi\right\}^{\frac{1}{2}(s-1)}},$$

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* See vol. IV. p. 24., First Series.

and we may now conveniently assume

$$2(f^2 + u'^2 + u^2) = h^2 + k^2,$$

$$2(f^2 + u'^2 - u^2)\cos\vartheta - 4uf\sin\vartheta\cos\phi$$

$$= 2\{(f^2 + u^2 - u'^2)^2 + 4u^2f^2\}^{\frac{1}{2}}\cos\theta = 2hk\cos\theta,$$

and $$\sin\phi\sin\vartheta = \sin\phi\sin\theta,$$

from which we deduce

$$h^2 = (u' + u)^2 + f^2, \quad k^2 = (u' - u)^2 + f^2,$$

$$\sin\vartheta\, d\phi\, d\vartheta = \sin\theta\, d\varphi\, d\theta\,;$$

the expression for U becomes

$$Uu = \tfrac{1}{2}H_{s-2}\int_0^{\pi}\int_0^{2\pi}\frac{\sin^{s-1}\theta\sin^{s-2}\varphi\, d\varphi\, d\theta}{(h^2-2hk\cos\theta-k^2)^{\frac{1}{2}(s-1)}}$$

$$= H_{s-1}\int_0^{\pi}\frac{\sin^{s-1}\theta\, d\theta}{(h^2 - 2hk\cos\theta + k^2)^{\frac{1}{2}(s-1)}}.$$

Let $$h\sin(\psi - \theta) = k\sin\psi\,;$$

by means of this transformation, observing that $h > k$, we readily find

$$Uu = \frac{H_s}{h^{s-1}}$$

or $$Uu = \frac{\pi^{\frac{1}{2}(s+1)}}{\Gamma\frac{1}{2}(s+1)}\,\frac{1}{\{\Sigma(x-x')^2 + (u+u')^2\}^{\frac{1}{2}(s-1)}},$$

which is the same as the result previously obtained.

St. Peter's College, Oct. 3, 1846.

ON CERTAIN FORMULÆ FOR DIFFERENTIATION, WITH APPLICATIONS TO THE EVALUATION OF DEFINITE INTEGRALS.

By Arthur Cayley.

In attempting to investigate a formula in the theory of multiple definite integrals (which will be noticed in the sequel), I was led to the question of determining the $(i+1)^{\text{th}}$ differential coefficient of the $2i^{\text{th}}$ power of $\sqrt{(x+\lambda)} - \sqrt{(x+\mu)}$; the only way that occurred for effecting this was to find the successive differential coefficients of this quantity, which may be effected as follows. Assume

$$U_{k,i} = \{(x+\lambda)(x+\mu)\}^{\frac{1}{2}k}\{\sqrt{(x+\lambda)} - \sqrt{(x+\mu)}\}^{2i},$$

then

$$\frac{1}{U_{k,i}}\frac{d}{dx}U_{k,i} = \tfrac{1}{2}k\frac{2x+\lambda+\mu}{(x+\lambda)(x+\mu)} - \frac{i}{\sqrt{\{(x+\lambda)(x+\mu)\}}}$$

$$= \tfrac{1}{2}k\frac{\{\sqrt{(x+\lambda)}+\sqrt{(x+\mu)}\}^2 - 2\sqrt{\{(x+\lambda)(x+\mu)\}}}{(x+\lambda)(x+\mu)} - \frac{i}{\sqrt{\{(x+\lambda)(x+\mu)\}}}$$

$$= \tfrac{1}{2}k\frac{(\lambda-\mu)^2}{\{\sqrt{(x+\lambda)}-\sqrt{(x+\mu)}\}^2(x+\lambda)(x+\mu)} - \frac{k+i}{\sqrt{\{(x+\lambda)(x+\mu)\}}}.$$

Or, attending to the signification of $U_{k,i}$,

$$\frac{d}{dx}U_{k,i} = \tfrac{1}{2}k(\lambda-\mu)^2 U_{k-2,i-1} - (k+i)U_{k-1,i}.$$

Hence $-\dfrac{1}{i}\dfrac{d}{dx}U_{0,i} = U_{-1,i}$

$$\frac{1}{i}\frac{d^2}{dx^2}U_{0,i} = \tfrac{1}{2}(\lambda-\mu)^2 U_{-s,i-1} + (i-1)U_{-2,i},$$

&c.

from which the law is easily seen to be of the form

$$\frac{(-)^r}{i}\left(\frac{d}{dx}\right)^r U_{0,i} = S_\theta K_{r,\theta}(\lambda-\mu)^{2r-2-2\theta}U_{-2r+1+\theta,\,i-r+1+\theta}$$

(where the extreme values of θ are 0 and $(r-1)$ respectively) and $K_{r,\theta}$ is determined by

$$K_{r+1,\theta+1} = (r-1-\tfrac{1}{2}\theta)K_{r,\theta+1} + (i-3r+2+2\theta)K_{r,\theta}.$$

This equation is satisfied by

$$K_{r,\theta} = \frac{\Gamma(r-\frac{1}{2}-\theta)\Gamma(2r-1-\theta)\Gamma(i-r+\theta+1)}{\Gamma\frac{1}{2}\Gamma(\theta+1)\Gamma(2r-1-2\theta)\Gamma(i-r+1)}.$$

For in the first place this gives

$$(r-1-\tfrac{1}{2}\theta)K_{r,\theta+1}$$

$$= \frac{(r-1-\frac{1}{2}\theta)}{\Gamma\frac{1}{2}\Gamma(\theta+2)}\,\frac{\Gamma(r-\frac{3}{2}-\theta)\Gamma(2r-2-\theta)\Gamma(i-r+\theta+2)}{\Gamma(2r-3-2\theta)\Gamma(i-r+1)}$$

$$= \frac{\Gamma(r-\frac{1}{2}-\theta)\Gamma(2r-1-\theta)\Gamma(i-r+\theta+2)}{\Gamma(\frac{1}{2})\Gamma(\theta+2)\Gamma(2r-2-2\theta)\Gamma(i-r+1)}.$$

And hence the second side of the equation reduces itself to

$$\frac{\Gamma(r-\frac{1}{2}-\theta)\Gamma(2r-1-\theta)\Gamma(i-r+\theta+1)}{\Gamma(\frac{1}{2})\Gamma(\theta+2)\Gamma(2r-1-2\theta)\Gamma(i-r+1)}$$

$$\{2(r-1-\theta)(i-r+\theta+1) + (\theta+1)(i-3r+2-2\theta)\},$$

where the quantity within brackets reduces itself to $(i-r)$ $(2r-1-\theta)$, so that the above value reduces itself to $K_{r+1,\theta+1}$, which verifies the equation in question. Also by comparing the first few terms, it is immediately seen that the above is the correct value of $K_{r,\theta}$, so that

$$\frac{(-)^r}{i}\left(\frac{d}{dx}\right)^r U_{0,i}$$

$$=S_\theta \frac{\Gamma(r-\frac{1}{2}-\theta)\,\Gamma(2r-1-\theta)\,\Gamma(i-r+\theta+1)}{\Gamma(\frac{1}{2})\,\Gamma(\theta+1)\,\Gamma(2r-1-\theta)\,\Gamma(i-r+1)}(\lambda-\mu)^{2r-2-\theta}\,U_{-2r+1-\theta,\,i-r+\theta+1} \quad \ldots\ldots(1),$$

θ extending as before from 0 to $(r-1)$. In particular if i be integer and $r=i+1$,

$$\frac{(-)^{i+1}}{i}\left(\frac{d}{dx}\right)^{i+1}\{\sqrt{(x+\lambda)}-\sqrt{(x+\mu)}\}^{2i}$$

$$=\frac{\Gamma(i+\frac{1}{2})}{\Gamma(\frac{1}{2})}(\lambda-\mu)^{2i}\frac{1}{\{(x+\lambda)(x+\mu)\}^{i+\frac{1}{2}}} \quad \ldots.(2),$$

(since the factor $\Gamma(i-r+\theta+1)\div\Gamma(i-r+1)$ vanishes except for $\theta=0$ on account of $\Gamma(i-r+1)=\infty$). Thus also, if r be greater than $(i+1)$, $=i+1+s$ suppose, then

$$(-)^s\left(\frac{d}{dx}\right)^s \frac{1}{\{(x+\lambda)(x+\mu)\}^{i+\frac{1}{2}}}$$

$$=S_\theta \frac{\Gamma(i+s+\frac{1}{2}-\theta)\,\Gamma(2i+2s+1-\theta)\,\Gamma(\theta-s)}{\Gamma(i+\frac{1}{2})\,\Gamma(\theta+1)\,\Gamma(2i+2s+1-2\theta)\,\Gamma(-s)}$$

$$(\lambda-\mu)^{2s-2\theta}\,U_{-2i-2s-1-\theta,\,-s+\theta} \quad \ldots(3),$$

where θ extends only from $\theta=0$ to $\theta=s$, on account of the factor $\Gamma(\theta-s)\div\Gamma(-s)$, which vanishes for greater values of θ: a rather better form is obtained by replacing this factor by $(-)^\theta \dfrac{\Gamma(1+s)}{\Gamma(1+s-\theta)}$.

The above formulæ have been deduced on the supposition of i being an integer; assuming that they hold generally, the equation (2) gives, by writing $(i-\frac{1}{2})$ for i,

$$\frac{(-)^{i+\frac{1}{2}}}{i-\frac{1}{2}}\left(\frac{d}{dx}\right)^{i+\frac{1}{2}}\{\sqrt{(x+\lambda)}-\sqrt{(x+\mu)}\}^{2i-1}$$

$$=\frac{\Gamma i}{\Gamma(\frac{1}{2})}(\lambda-\mu)^{2i-1}\frac{1}{\{(x+\lambda)(x+\mu)\}^{i}}.$$

Or integrating $(i+\frac{1}{2})$ times by means of the formula

$$\int_0^\infty x^{i-\frac{1}{2}} fx\,dx=\frac{\Gamma(i+\frac{1}{2})}{(-)^{i+\frac{1}{2}}}\left(\int_\infty da\right)^{i+\frac{1}{2}} fa, \quad a=0;$$

this gives

$$\int_0^\infty \frac{x^{i-\frac{1}{2}}dx}{\{(x+\lambda)(x+\mu)\}^i} = \frac{\Gamma\frac{1}{2}\,\Gamma(i-\frac{1}{2})}{\Gamma i}\,\frac{1}{(\sqrt{\lambda}+\sqrt{\mu})^{2i-1}} \quad \ldots (4),^*$$

whence also

$$\int_0^\infty \frac{x^{i-\frac{1}{2}}dx}{(x+\lambda)^{i+1}(x+\mu)^i} = \frac{\Gamma\frac{1}{2}\,\Gamma(i+\frac{1}{2})}{\Gamma i}\,\frac{1}{(\sqrt{\lambda}+\sqrt{\mu})^{2i}\sqrt{\lambda}} \quad \ldots (5).$$

And from these, by simple transformations,

$$\int_\beta^\alpha \frac{(\alpha-x)^{i-\frac{1}{2}}(x-\beta)^{i-\frac{1}{2}}dx}{\{(\alpha-x)+m(x-\beta)\}^i} = \frac{\Gamma\frac{1}{2}\,\Gamma(i+\frac{1}{2})}{\Gamma(i+1)}\,\frac{(\alpha-\beta)^i}{(\sqrt{m}+1)^{2i}} \quad \ldots (6),$$

$$\int_\beta^\alpha \frac{(\alpha-x)^{i-\frac{1}{2}}(x-\beta)^{i-\frac{3}{2}}dx}{\{(\alpha-x)+m(x-\beta)\}^i} = \frac{\Gamma\frac{1}{2}\,\Gamma(i-\frac{1}{2})}{\Gamma i}\,\frac{(\alpha-\beta)^{i-1}}{(\sqrt{m}+1)^{2i-1}} \ldots (7).$$

These last two formulæ are connected also by the following general property:

"If $\quad (a, b, i) = \int_\beta^\alpha \frac{(\alpha-x)^{a-1}(x-\beta)^{b-1}\,dx}{\{(\alpha-x)+m(x-\beta)\}^i},$

then $(a, b, i) = \frac{\Gamma a\,\Gamma b}{\Gamma(a+b-i)\,\Gamma i}.(\alpha-\beta)^{b-i}\,(a+b-i, i, b)$"...(8),

which I have proved by means of a double integral. From (6) we may obtain for $\gamma < 1$,

$$\int_{-1}^1 \frac{(1-x^2)^{i-\frac{1}{2}}dx}{(1-2\gamma x+\gamma^2)^i} = \frac{\Gamma\frac{1}{2}\,\Gamma(i+\frac{1}{2})}{\Gamma(i+1)} \quad \ldots\ldots\ldots\ldots (9),$$

which however is only a particular case of

$$\int_{-1}^1 dx\,(1-x^2)^{i-\frac{1}{2}}(1-2\gamma x+\gamma^2)^{-i}\frac{d}{d\beta}\left[\beta^i\left(1-2\frac{\beta}{\gamma}x+\frac{\beta^2}{\gamma^2}\right)^{-i}\right]$$

$$= \frac{\Gamma\frac{1}{2}\,\Gamma(i+\frac{1}{2})}{\Gamma(i+1)}\,\beta^{i-1}(1-\beta)^{-2i} \quad \ldots\ldots (10),$$

which supposes γ and $\frac{\beta}{\gamma}$ each less than unity. This formula was obtained in the case of $(i+\frac{1}{2})$ an integer, from a theorem, *Leg. Cal. Int.*, tom. II. p. 258, but there is no doubt that it is generally true.

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* This is immediately transformed into

$$\int_0^\infty \frac{x^{i-\frac{1}{2}}dx}{(ax^2+bx+c)^i} = \frac{\Gamma\frac{1}{2}\,\Gamma(i-\frac{1}{2})}{\Gamma i}\,\frac{1}{\{b+2\sqrt{(ac)}\}^{i-\frac{1}{2}}},$$

which is a particular case of a formula which will be demonstrated in a subsequent paper.

From (9), by writing $x = \cos\theta$, we have

$$\int_0^\pi \frac{\sin^{2i}\theta\, d\theta}{(1 - 2\gamma\cos\theta + \gamma^2)^i} = \frac{\Gamma\frac{1}{2}\,\Gamma(i+\frac{1}{2})}{\Gamma(i+1)} \quad \ldots\ldots (11),$$

which may also be demonstrated by the common equation in the theory of elliptic functions $\sin(\phi - \theta) = \gamma \sin\phi$, as was pointed out to me by Mr. Thomson. It may be compared with the following formula of Jacobi's, *Crelle*, tom. XV. p. 7.

$$\int_0^\pi \frac{\sin^{2i-1}\theta\, d\theta}{(1 - 2\gamma\cos\theta + \gamma^2)^i} = \frac{1}{\Gamma(i+\frac{1}{2})} \int_0^\pi \frac{\cos(i-\frac{1}{2})\,\theta\, d\theta}{\sqrt{(1 - 2\gamma\cos\theta + \gamma^2)}} \quad ..(12).$$

Consider the multiple integral

$$W = \int \frac{dx\, dy\, ..}{\{(x-a)^2 + \,..\, u^2\}^i} \quad \ldots\ldots\ldots\ldots (13),$$

the number of variables being $(2i + 1)$ (not necessarily odd), and the equation of the limits being

$$x^2 + y^2 \,.. = \xi.$$

Then, as will presently be shewn, W may be expanded in the form

$$W = \pi^{i+\frac{1}{2}}\, S_\lambda \frac{(-)^\lambda A^\lambda}{2^{2\lambda}\,\Gamma(\lambda+1)\,\Gamma(i+\lambda+\frac{1}{2})} \left(\frac{d}{du}\right)^{2\lambda} \int_0 \xi^{i-\frac{1}{2}} (\xi + u^2)^{-i} d\xi$$

$$\ldots\ldots (14),$$

where $A = a^2 + b^2 + \ldots.$ and λ extends from 0 to ∞. Suppose next

$$V = \int \frac{dx\, dy\, ..}{\{(x-a)^2 .. + u^2\}^i\, (x^2 + .. \,v^2)^{i+1}} \quad \ldots\ldots (15):$$

the number of variables as before, and the limits for each variable being $-\infty, \infty$. We have immediately

$$V = \int_0^\infty \frac{1}{(\xi + v^2)^{i+1}} \frac{dW}{d\xi}\, d\xi\,;$$

W as before, *i.e.*

$$V = \pi^{i+\frac{1}{2}} S_\lambda \frac{(-)^\lambda A^\lambda}{2^{2\lambda}\,\Gamma(\lambda+1)\,\Gamma(i+\lambda+\frac{1}{2})} \left(\frac{d}{du}\right)^{2\lambda} \int_0^\infty \frac{\xi^{i-\frac{1}{2}}\, d\xi}{(\xi+u^2)^i\,(\xi+v^2)^{i+1}}.$$

But writing $u^2\, v^2$ for λ, μ in the formula (5) (u and v being supposed positive), the integral in this formula is

$$\frac{\sqrt{\pi}\,\Gamma(i+\frac{1}{2})}{\Gamma(i+1)} \frac{1}{v\,(u+v)^{2i}}.$$

Hence, after a slight reduction,

$$V = \frac{\pi^{i+1}}{v\Gamma(i+1)} S \frac{(-)^\lambda \Gamma(i+\lambda+1)}{\Gamma(i+1)\Gamma(\lambda+1)} \frac{A^\lambda}{\{(u+v)^2\}^\lambda};$$

or finally $$V = \frac{\pi^{i+1}}{\Gamma(i+1)} \frac{1}{v\{(u+v)^2 + A\}^i} \quad \ldots\ldots\ldots(16),$$

a remarkable formula, the discovery of which is due to Mr. Thomson. It only remains to prove the formula for W. Out of the variety of ways in which this may be accomplished, the following is a tolerably simple one. In the first place, by a linear transformation corresponding to that between two sets of rectangular axes, we have

$$W = \int \frac{dx\,dy\,..}{\{(x - \sqrt{A})^2 + y^2 .. + u^2\}^i};$$

or expanding in powers of A, and putting for shortness $R = x^2 + y^2 \ldots + u^2$, the general term of W is

$$(-)^\sigma A^\lambda \frac{\Gamma(i+\lambda+\sigma)}{\Gamma i\, \Gamma(\lambda-\sigma+1)\Gamma(2\sigma+1)} 2^{2\sigma} \int x^{2\sigma} R^{-i-\lambda-\sigma} dx\,dy\,..$$

the limits being as before $x^2 + y^2 + \ldots = \xi$. To effect the integrations, write $\sqrt{\xi}\sqrt{x}$, $\sqrt{\xi}\sqrt{y}$, &c. for $x, y \ldots$ So that the equation of the limits becomes $x + y + \ldots = 1$. Also restricting the integral to positive values, we shall multiply it by 2^{2i+1}. The integral thus becomes

$$\xi^{\sigma+i+\frac{1}{2}} \int x^{\sigma-\frac{1}{2}} y^{-\frac{1}{2}} .. \{\xi(x+y..) + u^2)^{-i-\lambda-\sigma} dx\,dy.$$

Equivalent to

$$\xi^{\sigma+i+\frac{1}{2}} \frac{\Gamma(\sigma+\frac{1}{2})\pi^i}{\Gamma(i+\sigma+\frac{1}{2})} \int_0^1 \theta^{i+\sigma-\frac{1}{2}} (\xi\theta + u^2)^{-i-\lambda-\sigma} d\theta;$$

i.e. to $$\frac{\Gamma(\sigma+\frac{1}{2})\pi^i}{\Gamma(i+\sigma+\frac{1}{2})} \int_0 \xi^{i+\sigma-\frac{1}{2}} (\xi + u^2)^{-i-\lambda-\sigma} d\xi.$$

Or after a slight reduction, the general term of W is

$$\frac{\pi^{i+\frac{1}{2}}}{\Gamma i} (-)^{\lambda+\sigma} A^\lambda \frac{\Gamma(i+\lambda+\sigma)}{\Gamma(\sigma+1)\Gamma(\lambda-\sigma+1)\Gamma(i+\sigma+\frac{1}{2})} \int_0 \xi^{i+\sigma-\frac{1}{2}} (\xi + u^2)^{-i-\lambda-\sigma} d\xi,$$

where σ may be considered as extending from 0 to λ inclusively, and then λ from 0 to ∞. But by a formula easily proved

$$\left(\frac{d}{du}\right)^{2\lambda} (\xi + u^2)^{-i} = \frac{2^{2\lambda}\Gamma(\lambda+1)\Gamma(i+\lambda+\frac{1}{2})}{\Gamma i}$$

$$S(-)^\sigma \frac{\Gamma(i+\lambda+\sigma)}{\Gamma(\sigma+1)\Gamma(\lambda-\sigma+1)\Gamma(i+\sigma+\frac{1}{2})} \xi^\sigma (\xi + u^2)^{-i-\lambda-\sigma},$$

where σ extends from 0 to λ. Hence substituting, and prefixing the summatory sign

$$W = \pi^{i+\frac{1}{2}} S \frac{(-)^\lambda A^\lambda}{2^{2\lambda}\,\Gamma(\lambda+1)\,\Gamma(i+\lambda+\frac{1}{2})} \left(\frac{d}{du}\right)^{2\lambda} \int_0 \xi^{i-\frac{1}{2}} (\xi + u^2)\, d\xi,$$

where λ extends from 0 to ∞, the formula required.

ON THE CAUSTIC BY REFLECTION AT A CIRCLE.

By Arthur Cayley.

The following solution of the problem is that given by M. de St. Laurent (*Annales de Gergonne*, tom. XVII. p. 128); the process of elimination is somewhat different.

The centre of the circle being taken for the origin, let k be its radius; a, b the coordinates of the luminous point; ξ, η those of the point at which the reflection takes place; x, y of any point in the reflected ray: we have in the first place

$$\xi^2 + \eta^2 = k^2 \quad \ldots\ldots\ldots\ldots\ldots\ldots (1).$$

There is no difficulty in finding the equation of the reflected ray*

$$(b\xi - a\eta)(\xi x + \eta y - k^2) + (y\xi - \xi\eta)(a\xi + b\eta - k^2) = 0.$$

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* To do this in the simplest way, write

$$\rho^2 = (\xi - x)^2 + (\eta - y)^2, \quad \sigma^2 = (\xi - a)^2 + (\eta - b)^2.$$

Then, by the condition of reflection,

$$\rho + \sigma = \text{min.},$$

ρ, σ being considered as functions of the variables ξ, η, which are connected by the equation (1). Hence

$$\frac{\xi - x}{\rho} + \frac{\xi - a}{\sigma} + \lambda\xi = 0,$$

$$\frac{\eta - y}{\rho} + \frac{\eta - b}{\sigma} + \lambda\eta = 0.$$

Or eliminating λ,

$$\frac{\eta x - \xi y}{\rho} + \frac{\eta a - \xi b}{\sigma} = 0,$$

whence

$$(\eta x - \xi y)^2\,[(\xi - a)^2 + (\eta - b)^2] = (\eta a - \xi b)^2\,[(\xi - x)^2 + (\eta - y)^2].$$

Or

$$\{(\eta x - \xi y)(\xi - a) - (\eta a - \xi b)(\xi - x)\}\,[(\eta x - \xi y)(\xi - a) + (\eta a - \xi b)(\xi - x)]$$
$$+ \{(\eta x - \xi y)(\eta - b) - (\eta a - \xi b)(\eta - y)\}\,[(\eta x - \xi y)(\eta - b) + (\eta a - \xi b)(\eta - y)] = 0.$$

The factors in { } reduce themselves respectively to ξP and ηP, where $P = \xi(b - y) - \eta(a - x) + ay - bx$, omitting the factor P, (which equated to zero, is the equation of the line through (a, b) and (ξ, η).) And replacing $\xi(\xi - a) + \eta(\eta - b)$ and $\xi(\xi - x) + \eta(\eta - y)$ by $k^2 - a\xi - b\eta$ and $k^2 - \xi x - \eta y$, respectively, we have the equation given above.

Or, arranging the terms in a more convenient order,

$$(bx+ay)(\xi^2-\eta^2)+2(by-ax)\xi\eta-k^2(b+y)\xi+k^2(a+x)\eta=0\ldots(2).$$

Hence, considering ξ, η as indeterminate parameters connected by the equation (1), the locus of the curve generated by the continued intersections of the lines (2) will be found by eliminating ξ, η, λ from these equations and the system

$$\xi[\lambda+2(bx+ay)]+\eta[2(by-ax)]-k^2.(b+y)=0\ldots(3),$$
$$\xi[2(by-ax)]+\eta[\lambda-2(bx+ay)]+k^2.(a+x)=0\ldots(4),$$

whence, multiplying by ξ, η, adding and reducing by (2), we have

$$-\xi(b+y)+\eta.(a+x)-\lambda=0\ldots\ldots\ldots(5),$$

which replaces the equation (2) or (2′). Or the equations from which ξ, η, λ are to be eliminated are (1), (3), (4), (5).

From (3), (4), (5), by the elimination of ξ, η, we have

$$\begin{aligned}-\lambda\{\lambda^2-4(bx+ay)^2\}&-4k^2.(by-ax)(a+x)(b+y)\\ &-k^2.(a+x)^2.[\lambda+2(bx+ay)]\\ &-k^2.(b+y)^2.[\lambda-2(bx+ay)]\\ &+4\lambda(by-ax)^2=0\ldots\ldots\ldots(6).\end{aligned}$$

Or, reducing,

$$\begin{aligned}-\lambda^3+\lambda.\{4(a^2+b^2)(x^2+y^2)-k^2.[(a+x)^2+(b+y)^2]\}\\ -2k^2.(bx-ay)(x^2+y^2-a^2-b^2)=0\ldots\ldots(7);\end{aligned}$$

which may be represented by

$$-\lambda^3+\lambda Q-2R=0\ldots\ldots\ldots(7').$$

Again, from the equations (4), (3), transposing the last terms and adding the squares, also reducing by (1),

$$\begin{aligned}k^4.[(a+x)^2+(b+y)^2]=k^2\lambda^2+4k^2.(a^2+b^2)(x^2+y^2)\\ +4\lambda.[(\xi^2-\eta^2)(bx+ay)+2\xi\eta(by-ax)]\ldots(8).\end{aligned}$$

But from the same equations, multiplying by ξ, η and adding, also reducing by (1),

$$\begin{aligned}k^2\lambda+2(bx+ay)(\xi^2-\eta^2)+4\xi\eta(by-ax)\\ +k^2[-\xi(b+y)+\eta(a+x)]=0\ldots\ldots(9).\end{aligned}$$

Or reducing by (5) and dividing by two,

$$k^2\lambda+(bx+ay)(\xi^2-\eta^2)+2\xi\eta(by-ax)=0\ldots(10).$$

Using this to reduce (8),

$$k^2.[(a+x)^2+(b+y)^2]=4(a^2+b^2)(x^2+y^2)-3\lambda^2\ldots(11).$$

Or, from the value of P,

$$-3\lambda^2+Q=0\ldots\ldots\ldots(12),$$

which singularly enough is the derived equation of (7′) with respect to λ: so that the equation of the curve is obtained by expressing that two of the roots of the equation (7′) are equal. Multiplying (12) by λ and reducing by (7′),

$$-\lambda Q + 3R = 0.$$

Or combining this with (12),

$$27R^2 - Q^3 = 0.$$

Or replacing R, Q by their values,

$$27k^4.(bx - ay)^2.(x^2 + y^2 - a^2 - b^2)^2$$
$$- \{4(a^2 + b^2)(x^2 + y^2) - k^2.[(a + x)^2 + (b + y)^2]\}^3 = 0,$$

the equation of M. de St. Laurent.

ON SYMBOLICAL GEOMETRY.

By Sir William Hamilton.

(*Continued from* p. 52.)

Symbolical Expressions for a Cyclic Cone; Relations of such a Cone, and of its Cyclic Planes, to a Product of two Geometrical Fractions.

20. It is evidently a determinate* problem to construct a *cyclic cone*, that is, a cone with circular base (called usually a cone of the second degree), when three of the *sides* (or generating straight lines) of the cone are given in position, and when the plane of the base is parallel to a given *cyclic plane*, which passes through the vertex. To treat this problem, which may be regarded as a fundamental one in the theory of such cones, by a method derived from the principles of the foregoing articles, let the three given sides be denoted by the letters a, b, c; and let the two known lines, in which the given cyclic plane is cut by the planes of the two pairs,

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* The evident and known determinateness of this problem, corresponding to that of the elementary problem of circumscribing a circle about a given plane triangle, was tacitly assumed, but might with advantage have been expressly referred to, in the outline of a demonstration which was given in the note to Art. (18). The reasoning, towards the end of that note, would then stand thus:—If D be any fourth point on the determined spherical conic, which passes through the three points A, B, C, and has the arc A′B′ for a cyclic arc, it is also a fourth point on the determined spherical conic which passes through the same three points and has the arc B″C″ for a cyclic arc; therefore the two conics, determined by these two sets of conditions, coincide one with the other: or, in other words, the arc B″C″ is a *second* cyclic arc of the *same* spherical conic, of which the arc A′B′ is a *first* cyclic arc.

ab and bc, be denoted by a′ and b′; also let d denote any fourth side of the sought cyclic cone, and c′, d′ the lines of intersection of the given cyclic plane with the variable planes of cd and da; then, if suitable lengths be assigned to these straight lines, of which the relative *directions* in space are the chief object of the present investigation, the following equality between two products of certain geometrical fractions will exist, and may be regarded as a form of the *equation of the cone:*

$$\frac{\mathrm{c}}{\mathrm{b}'}\,\frac{\mathrm{a}'}{\mathrm{a}} = \frac{\mathrm{c}}{\mathrm{c}'}\,\frac{\mathrm{d}'}{\mathrm{a}} \ldots\ldots\ldots\ldots\ldots (147).$$

That is to say, when this equation is satisfied, the two lines which are the respective intersections of the planes of the fractional factors of these two equal products, namely the intersection b of the planes aa′ and b′c, and the intersection d of the planes ad′ and c′c, are two sides of a cyclic cone, which has for two other sides the lines a and c, and which has for one cyclic plane the common plane of the four lines a′, b′, c′, and d′; these eight lines, a, b, c, d, a′, b′, c′, d′, being here supposed to diverge from one common origin, namely the vertex (or centre) of the cone. This may easily be shown to be a consequence of what has been already established, respecting the connexion of the cyclic arcs of a spherical conic with the symbolic sums of certain other arcs. Or, without introducing any sphere, we may observe that, by (121) and its converse, the equation (147) may be abridged to the following:

$$\frac{\mathrm{a}'}{\mathrm{b}'} = \frac{\mathrm{d}'}{\mathrm{c}'}; \quad \text{or,} \quad \frac{\mathrm{a}'}{\mathrm{b}'}\,\frac{\mathrm{c}'}{\mathrm{d}'} = 1 \ldots\ldots\ldots (148);$$

which shows, in virtue of the notation here employed, that besides a certain proportionality of lengths, not necessary now to be considered, there exists an equality between the angles of rotation, in one common plane, which would transport the lines b′ and c′, respectively, into the directions of a′ and d′. But the four lines a′, b′, c′, d′ are respectively parallel to the four symbolic differences, b − a, c − b, d − c, a − d, or to the four straight lines BA, CB, DC, AD, that is to the successive sides of the plane quadrilateral ABCD, if we now suppose the lines a, b, c, d to terminate, in the points A, B, C, D, on a transversal plane parallel to the plane of a′ b′ c′ d′. We may therefore present the relation (148) under either of the two forms:

$$\frac{\mathrm{b}-\mathrm{a}}{\mathrm{c}-\mathrm{b}}\,\frac{\mathrm{d}-\mathrm{c}}{\mathrm{a}-\mathrm{d}} = x; \quad \text{or} \quad \frac{\mathrm{BA}}{\mathrm{CB}}\,\frac{\mathrm{DC}}{\mathrm{AD}} = x \ldots (149);$$

in which x is a positive or negative scalar; or, using the characteristic V of the operation of taking the vector part, we may write:

$$\mathrm{V}.\frac{\mathrm{b}-\mathrm{a}}{\mathrm{c}-\mathrm{b}}\,\frac{\mathrm{d}-\mathrm{c}}{\mathrm{a}-\mathrm{d}}=0\,;\quad\text{or}\quad \mathrm{V}.\frac{\mathrm{BA}}{\mathrm{CB}}\,\frac{\mathrm{DC}}{\mathrm{AD}}=0\,..(150).$$

When the scalar x is positive, then, by considering the two rotations above mentioned, we easily perceive that the two points B and D are at one common side of the straight line AC, and that this line subtends equal angles at those two points; being in one common plane with them, as indeed the second equation (149) sufficiently expresses, since it gives

$$\mathrm{V}\,\frac{\mathrm{BA}}{\mathrm{CB}}=x\mathrm{V}\,\frac{\mathrm{DA}}{\mathrm{CD}}\ \ldots\ldots\ldots\ (151);$$

so that the two triangles ABC, ADC, on the common base AC, have one common perpendicular to their planes, which must therefore coincide with each other. In the contrary case, namely when x is negative, the equation (151) still shows that the four points are (as above) coplanar with each other; and while the points B and D are now at opposite sides of the line AC, the angles which this line subtends at those two points are now not equal but supplementary. In each case, therefore, the four points ABCD are on the circumference of one common circle; the four lines a, b, c, d are consequently sides of a cyclic cone; and the plane of the four other lines a′, b′, c′, d′ is a cyclic plane of that cone.

21. In the foregoing article, the coplanarity of each of the four sets of three lines, a′ab, b′bc, c′cd, d′da, allows us to suppose that four other lines b″, c″, d″, a″, in the same four planes respectively, and all, like the eight former lines, diverging from the vertex of the cone, are determined so as to satisfy the four equations:

$$\frac{\mathrm{b}''}{\mathrm{b}}=\frac{\mathrm{a}}{\mathrm{a}'};\quad \frac{\mathrm{c}''}{\mathrm{c}}=\frac{\mathrm{b}}{\mathrm{b}'};\quad \frac{\mathrm{d}''}{\mathrm{d}}=\frac{\mathrm{c}}{\mathrm{c}'};\quad \frac{\mathrm{a}''}{\mathrm{a}}=\frac{\mathrm{d}}{\mathrm{d}'}\ \ldots (152);$$

and then, since these equations, combined with (148), give, by the associative property of the multiplication of geometrical fractions, this other equation,

$$\frac{\mathrm{b}''}{\mathrm{c}''}=\frac{\mathrm{a}''}{\mathrm{d}''}\ \ldots\ldots\ldots\ldots\ (153),$$

it follows that these four new lines are in one common plane; and also that the rotations in that plane, from b″ and c″ to

a'' and d'', respectively, are equal. And this new plane is evidently a *second* cyclic plane of the same cone;* for we may now write, instead of (147), the analogous equation:

$$\frac{c}{c''}\,\frac{b''}{a} = \frac{c}{d''}\,\frac{a''}{a} \ldots\ldots\ldots\ldots (154);$$

the two members being here equal respectively to the reciprocals of the two members of the first equation (148): nor is it necessary to retain the restriction that the lines a, b, c, d should terminate in one common plane. In like manner, the two members of the equation (147) are respectively equal to the reciprocals of the two members of the equation (153); a geometrical (like an arithmetical) fraction being said to be changed to its *reciprocal,* when the numerator and denominator are interchanged. We have therefore this theorem:—*A cyclic cone is the locus of the intersection of the planes of two geometrical fractions, of which the product is a constant fraction, while the numerator of the multiplier and the denominator of the multiplicand are constant lines. These two lines are two fixed sides of the cone; the plane of the two other and variable lines,* which enter as denominator and numerator into the expressions of the same two fractional factors, *is one cyclic plane of that cone;* and *the plane of the constant product is the other cyclic plane.* The investigation in the last article shows also that the condition for four points ABCD being *concircular* or *homocyclic,* that is, for their being corners of a quadrilateral inscribed in a circle, is expressed by the second equation (150); which may therefore be called the *equation of homocyclicism.* The same investigation shows that if we only know that ABCD are four points on one common plane, we may still write an equation of the form (151); which may for that reason be said to be a *formula of coplanarity.*

[*To be continued.*]

ADDITIONAL CORRECTIONS FOR THE PRECEDING PORTION OF THIS PAPER.

In Note to Art. (8), p. 138 (vol. I.), *for* CB *read* CA.
In Art. (17), p. 263, the second spherical hexagon should be
$A''A'''\,B''B'''\,C''C'''$.
In Art. (18), p. 48, line 17 (vol. II.), *for* alteration *read* alternation.

* See the remarks made in the note to the foregoing article.

ON CERTAIN SYMBOLICAL REPRESENTATIONS OF FUNCTIONS.

By the Rev. Brice Bronwin.

A symbolical representation of Taylor's theorem has long been in use, and has been employed in integration. Sir John Herschel's and Sir William Hamilton's theorems may be thus represented. And there are some others of the kind, which I suppose to be new, and which it is the object of this paper to exhibit.

Unless it be otherwise stated, D always stands for $\frac{d}{do}$. Taylor's theorem is $\phi(x) = \varepsilon^{xD}\phi(o)$. Change $\phi(x)$ into $\phi(\varepsilon^x)$, and it becomes $\phi(\varepsilon^x) = \varepsilon^{xD}\phi(\varepsilon^o)$. Now change ε^x into x, and we have

$$\phi(x) = x^D\phi(\varepsilon^o) \dots\dots\dots\dots\dots (a).$$

This will serve to expand by the powers of log x; thus

$$\phi(x) = \phi(\varepsilon^o) + \frac{\log x}{1} D\phi(\varepsilon^o) + \frac{(\log x)^2}{1.2} D^2\phi(\varepsilon^o) + \&c.$$

We may multiply (a) by a function of x, and integrate. For

$$\int\phi(x)\,x^{n-1}dx = \frac{x^{n+D}}{n+D}\,\phi(\varepsilon^o) = \Sigma\left\{\frac{(lx)^m}{1.2\dots m}(n+D)^{m-1}\phi(\varepsilon^o)\right\}$$

$$= \Sigma\left[\frac{(lx)^m}{1.2\dots m}\,\varepsilon^{-no}D^{m-1}\{\varepsilon^{no}\phi(\varepsilon^o)\}\right] = \Sigma\left[\frac{(lx)^m}{1.2\dots m}\,D^{m-1}\{\varepsilon^{no}\phi(\varepsilon^o)\}\right].$$

But, by the theorem itself,

$$\int\phi(x)\,x^{n-1}dx = x^D\int\phi(\varepsilon^o)\,\varepsilon^{no}do$$

$$= \Sigma\left\{\frac{(lx)^m}{1.2\dots m}D^m\int\phi(\varepsilon^o)\,\varepsilon^{no}do\right\} = \Sigma\left[\frac{(lx)^m}{1.2\dots m}\,D^{m-1}\{\varepsilon^{no}\phi(\varepsilon^o)\}\right],$$

the same as before. To abridge, lx has been put for log x. Now, giving to n any values whatever, and an infinity of different ones, multiplying the results by any constants, and taking the sum of all the products, we have

$$\int\phi(x)f(x)\,dx = \left\{\int f(x)\,x^D dx\right\}\phi(\varepsilon^o).$$

It may be proved in the same manner, that Taylor's theorem may be multiplied by $f(x)$, and integrated; but the above includes the proof of it.

In (a) change $\phi(x)$ into $\phi\left(\frac{1}{x}\right)$, and we have $\phi\left(\frac{1}{x}\right) = x^D\phi(\varepsilon^{-o})$.

Change in this last x into $\frac{1}{x}$, and there results

$$\phi(x) = x^{-D}\phi(\varepsilon^{-o}) \dots\dots\dots\dots\dots (b).$$

In (a) change x into $a - x$, and in (b) into $a + x$; these formulæ become

$$\phi(a-x) = (a-x)^{D}\phi(\varepsilon^{o}), \quad \text{and} \quad \phi(a+x) = (a+x)^{-D}\phi(\varepsilon^{-o});$$

whence we derive

$$\int_0^a \phi(a-x)\, x^{n-1}dx = \left\{\int_0^a (a-x)^{D} x^{n-1} dx\right\} \phi(\varepsilon^{o})$$

$$= \frac{\Gamma(n)\,\Gamma(D+1)}{\Gamma(D+n+1)}\, a^{D+n}\phi(\varepsilon^{o}) = \frac{\Gamma(n)\, a^{D+n}}{(D+1)(D+2)\ldots(D+n)}\,\phi(\varepsilon^{o}).$$

But
$$\int \phi(a)\, da = \frac{a^{D+1}}{D+1}\,\phi(\varepsilon^{o}),$$

$$\int^2 \phi(a)\, da^2 = \frac{a^{D+2}}{(D+1)(D+2)}\,\phi(\varepsilon^{o}),$$

$$\int^n \phi(a)\, da^n = \frac{a^{D+n}}{(D+1)(D+2)\ldots(D+n)}\,\phi(\varepsilon^{o}).$$

Therefore
$$\int_0^a \phi(a-x)\, x^{n-1}dx = \Gamma(n) \int^n \phi(a)\, da^n,$$

the second member to be integrated from $a = o$ to $a = a$, as the first vanishes when $a = o$.

$$\int_0^\infty \phi(a+x)\, x^{n-1}dx = \left\{\int_0^\infty (a+x)^{-D} x^{n-1} dx\right\} \phi(\varepsilon^{-o})$$

$$= \frac{\Gamma(n)\,\Gamma(D-n)}{\Gamma(D)}\, a^{-D+n}\phi(\varepsilon^{-o}) = \frac{\Gamma(n)\, a^{-D+n}}{(D-1)(D-2)\ldots(D-n)}\,\phi(\varepsilon^{-o}).$$

$$\int \phi(a)\, da = -\frac{a^{-D+1}}{D-1}\,\phi(\varepsilon^{-o}),$$

$$\int^2 \phi(a)\, da^2 = (-1)^2 \frac{a^{-D+2}}{(D-1)(D-2)}\,\phi(\varepsilon^{-o}),$$

$$\int^n \phi(a)\, da^n = (-1)^n \frac{a^{-D+n}}{(D-1)(D-2)\ldots(D-n)}\,\phi(\varepsilon^{-o}).$$

Therefore
$$\int_0^\infty \phi(a+x)\, x^{n-1}dx = (-1)^n\, \Gamma(n) \int^n \phi(a)\, da^n,$$

the second member to be integrated from $a = A$ to $a = a$, A being the value of a which makes the first member vanish.

I cannot stop to comment on the several steps of these two examples, which are known,* and are only given here by

* See vol. I. p. 114, First Series.

way of illustration. I now subjoin the integration of a differential equation, which may be more conveniently effected otherwise; but it may be well to shew that (*a*) may in some cases be thus applied.

Let
$$x^2 \frac{d^2y}{dx^2} + mx \frac{dy}{dx} + ny = 0.$$

Make $y = \phi(x) = x^D \phi(\varepsilon^o)$. With this value of y, the proposed equation becomes
$$x^D \left\{D^2 + (m-1) D + n\right\} \phi(\varepsilon^o) = 0.$$

Or, as $\phi(\varepsilon^o)$ is independent of x,
$$\left\{D^2 + (m-1) D + n\right\} \phi(\varepsilon^o) = o,$$
the integral of which is $\phi(\varepsilon^o) = A\varepsilon^{b_1 o} + B\varepsilon^{b_2 o}$, b_1, b_2 being the roots of $b^2 + (m-1) b + n = o$. Therefore
$$y = \phi(x) = Ax^{b_1} + Bx^{b_2}.$$

In (*a*) change $\phi(x)$ into $\phi(\varepsilon^x)$; then $\phi(\varepsilon^x) = x^D \phi(\varepsilon^o)$. Change in this ε^x into x, or x into $\log x = lx$; we have
$$\phi(x) = (lx)^D \phi(\varepsilon^{\varepsilon^o}) \;\ldots\ldots\ldots\ldots\; (c).$$

This serves to develop $\phi(x)$ by the powers of $\log\log x = llx = l^2x$. We might apply it to integration. By a continued repetition of the same steps, we may find a formula to develop by the powers of $l^n x$.

We may treat $\phi(x) = (1+\Delta)^x \phi(o)$ in like manner; first changing $\phi(x)$ into $\phi(\varepsilon^x)$, then x into lx. Thus we should find
$$\left.\begin{aligned}\phi(x) &= (1+\Delta)^{lx}\phi(\varepsilon^o)\\ \phi(x) &= (1+\Delta)^{llx}\phi(\varepsilon^{\varepsilon^o})\end{aligned}\right\} \;\ldots\ldots\ldots\; (d).$$

The following is proved by developing the second member.
$$\frac{d^n\phi(x)}{dx^n} = \phi(x+D)\, o^n \;\ldots\ldots\ldots\ldots\; (e).$$

This gives
$$\frac{d^n\phi(o)}{do^n} = \phi(D)\, o^n.$$

Therefore
$$\phi(x) = \phi(o) + \frac{x}{1}\frac{d\phi(o)}{do} + \frac{x^2}{1.2}\frac{d^2\phi(o)}{do^2} + \&c.$$
$$= \phi(D)\left\{1 + \frac{ox}{1} + \frac{o^2x^2}{1.2} + \&c.\right\}.$$

Or
$$\phi(x) = \phi(D)\, \varepsilon^{ox} \;\ldots\ldots\ldots\ldots\; (f).$$

This might be thus investigated:
$$D\varepsilon^{ox} = x\varepsilon^{ox} = x, \quad D^2\varepsilon^{ox} = x^2, \quad D^n\varepsilon^{ox} = x^n.$$

Give to n an infinity of different values, which we may suppose to be either integer or fractional. Multiply each of the results by a constant, and take the sums of both members, which will be $\phi(D)\,\varepsilon^{ox}$ and $\phi(x)$, and which will thus have a more general form. But if the simple operation D be performed without adding any correction, we have $D^{-n}\varepsilon^{ox} = x^{-n}$; and thus negative exponents may be included, so that $\phi(x)$ may include all the forms with which we are acquainted, even $\log x$.

As D is here separated from x, and operates only upon o; we may, according to long established principles, multiply (f) by a function of x, and integrate. This formula may supply the place of Abel's definite integral $\int \varepsilon^{vx} f(v)\,dv$.

In (f) change x into $-x$, then $\phi(x)$ into $\phi(-x)$; we thus obtain

$$\phi(x) = \phi(-D)\,\varepsilon^{-ox} \ldots\ldots\ldots\ldots (g).$$

By $f(r) = (1+\Delta)^r f(o) = E^r f(o)$, the last may be thus exhibited,

$$\phi(x) = E^r \phi(-D)\,\varepsilon^{-rx} \ldots\ldots\ldots\ldots (h),$$

where D now denotes $\dfrac{d}{dr}$, and $E = 1 + \Delta$.

We will now give some examples of integration. In those which we shall select, we prefer employing the formula (h).

$$\int_0^\infty dv \cos uv\, \phi(v) = E^r \phi(-D) \int_0^\infty \varepsilon^{-rv} dv \cos uv = E^r \phi(-D) \frac{r}{r^2+u^2}.$$

$$\int_0^\infty\!\!\int_0^\infty du\,dv \cos au \cos uv\, \phi(v) = E^r \phi(-D) \int_0^\infty \frac{r\,du \cos au}{r^2+u^2}$$

$$= \tfrac{1}{2}\pi\, E^r \phi(-D)\,\varepsilon^{-ra} = \tfrac{1}{2}\pi\, \phi(a).$$

$$\int_0^\infty dv \sin uv\, \phi(v) = E^r \phi(-D) \int_0^\infty \varepsilon^{-rv} dv \sin uv = E^r \phi(-D) \frac{u}{r^2+u^2}.$$

$$\int_0^\infty\!\!\int_0^\infty du\,dv \sin au \sin uv\, \phi(v) = E^r \phi(-D) \int_0^\infty \frac{u\,du \sin au}{r^2+u^2}$$

$$= \tfrac{1}{2}\pi\, E^r \phi(-D)\,\varepsilon^{-ra} = \tfrac{1}{2}\pi\, \phi(a).$$

These two double integrals, which are known, give Fourier's theorem.

Reverting now to (f), change $\phi(x)$ into $\phi(\varepsilon^x)$; and it becomes $\phi(\varepsilon^x) = \phi(\varepsilon^D)\,\varepsilon^{ox}$. But $\varepsilon^D = 1 + \Delta = E$. Therefore

$$\phi(\varepsilon^x) = \phi(E)\,\varepsilon^{ox} \ldots\ldots\ldots\ldots (i),$$

which is Herschel's theorem. Change x into $x\sqrt{(-1)}$, and into $-x\sqrt{(-1)}$, and add and subtract the results; we thus obtain

$$\tfrac{1}{2}\{\phi(\varepsilon^{x\sqrt{(-1)}}) + \phi(\varepsilon^{-x\sqrt{(-1)}})\} = \phi(E)\cos ox,$$

$$\frac{1}{2\sqrt{(-1)}}\{\phi(\varepsilon^{x\sqrt{(-1)}}) - \phi(\varepsilon^{-x\sqrt{(-1)}})\} = \phi(E)\sin ox,$$

$$\frac{1}{2}\int_0^\infty \frac{dx}{1+x^2}\{\phi(\varepsilon^{x\sqrt{(-1)}}) + \phi(\varepsilon^{-x\sqrt{(-1)}})\} = \phi(E)\int_0^\infty \frac{dx\cos ox}{1+x^2}$$
$$= \tfrac{1}{2}\pi\,\phi(E)\,\varepsilon^{-o} = \tfrac{1}{2}\pi\,\phi(\varepsilon^{-1}),$$

$$\frac{1}{2\sqrt{(-1)}}\int_0^\infty \frac{x\,dx}{1+x^2}\{\phi(\varepsilon^{x\sqrt{(-1)}}) - \phi(\varepsilon^{-x\sqrt{(-1)}})\} = \phi(E)\int_0^\infty \frac{x\,dx\sin ox}{1+x^2}$$
$$= \tfrac{1}{2}\pi\,\phi(E)\,\varepsilon^{-o} = \tfrac{1}{2}\pi\,\phi(\varepsilon^{-1}) \ldots\ldots\ldots (k).$$

Make $\phi(\varepsilon^{x\sqrt{(-1)}}) + \phi(\varepsilon^{-x\sqrt{(-1)}}) = \dfrac{\varepsilon^{ax\sqrt{(-1)}} + \varepsilon^{-ax\sqrt{(-1)}}}{\varepsilon^{bx\sqrt{(-1)}} + \varepsilon^{-bx\sqrt{(-1)}}} = \dfrac{\cos ax}{\cos bx}$

$$\phi(\varepsilon^{x\sqrt{(-1)}}) - \phi(\varepsilon^{-x\sqrt{(-1)}}) = \frac{\varepsilon^{ax\sqrt{(-1)}} - \varepsilon^{-ax\sqrt{(-1)}}}{\varepsilon^{bx\sqrt{(-1)}} + \varepsilon^{-bx\sqrt{(-1)}}} = \sqrt{(-1)}\,\frac{\sin ax}{\cos bx}.$$

Here a must be less than b, or the second members would reduce to a different form. These functional equations solved give

$$\phi(\varepsilon^{x\sqrt{(-1)}}) = \tfrac{1}{2}\,\frac{\varepsilon^{ax\sqrt{(-1)}} + \varepsilon^{-ax\sqrt{(-1)}}}{\varepsilon^{bx\sqrt{(-1)}} + \varepsilon^{-bx\sqrt{(-1)}}}$$

for the first, and

$$\phi(\varepsilon^{p\sqrt{(-1)}}) = \tfrac{1}{2}\,\frac{\varepsilon^{ax\sqrt{(-1)}} - \varepsilon^{-ax\sqrt{(-1)}}}{\varepsilon^{bx\sqrt{(-1)}} + \varepsilon^{-bx\sqrt{(-1)}}}$$

for the second. With these values of $\phi(\varepsilon^{x\sqrt{(-1)}})$, (k) give immediately

$$\int_0^\infty \frac{dx}{1+x^2}\,\frac{\cos ax}{\cos bx} = \tfrac{1}{2}\pi\,\frac{\varepsilon^a + \varepsilon^{-a}}{\varepsilon^b + \varepsilon^{-b}}\quad \int_0^\infty \frac{x\,dx}{1+x^2}\,\frac{\sin ax}{\cos bx} = -\tfrac{1}{2}\pi\,\frac{\varepsilon^a - \varepsilon^{-a}}{\varepsilon^b + \varepsilon^{-b}}.$$

In (i) change ε^x into x, and it gives

$$\phi(x) = \phi(E)\,x^o \ldots\ldots\ldots\ldots\ldots (l),$$

which is Hamilton's theorem.

Change in (l) $\phi(x)$ into $\phi(\varepsilon^x)$, and then x into $\log x$; there results

$$\phi(x) = \phi(\varepsilon^E)\,(lx)^o \ldots\ldots\ldots\ldots (m).$$

By a repetition of the same steps, we might add more theorems; and let it be remembered that we can always replace o by r, if it appear desirable to do so.

Let $\phi(a, x) = \Sigma(A_n x^n)$; then $\left(x\dfrac{d}{dx}\right)^r \phi(a, x) = \Sigma(n^r A_n x^n)$, and consequently $\psi\left(x\dfrac{d}{dx}\right)\phi(a, x) = \Sigma\{\psi(n)\,A_n x^n\}$. Make

$\psi(n) = \left(\frac{d}{da}\right)^n$, then $\left(\frac{d}{da}\right)^{x\frac{d}{dx}} \phi(a, x) = \Sigma \left\{\left(\frac{d}{da}\right)^n A_n x^n\right\}$. Suppose, now, $\phi(a, x) = \phi(a)\, \varepsilon^x$, and therefore $A_n = \frac{\phi(a)}{1.2\ldots n}$; the last becomes

$$\left(\frac{d}{da}\right)^{x\frac{d}{dx}} \{\phi(a)\, \varepsilon^x\} = \Sigma\left(\frac{d^n\phi(a)}{da^n}\; \frac{x^n}{1.2\ldots n}\right) = \phi(a + x).$$

Or
$$\phi(a + x) = \left(\frac{d}{da}\right)^{x\frac{d}{dx}} \{\phi(a)\, \varepsilon^x\} \ldots\ldots\ldots (n).$$

This appears a mere curiosity; but we do not know what may prove useful. The preceding theorems will of course apply, if there be more variables than one. Thus we shall have

$$\phi(x, y) = x^D y^{D'} \phi(\varepsilon^o, \varepsilon^{o'}), \quad \phi(x, y) = \phi(D, D')\, \varepsilon^{ox + o'y},$$

$\phi(x, y) = \phi(E, E')\, x^o y^{o'}$, &c.; and similarly for more variables. It must be observed, that D and E operate upon o, D' and E' upon o'.

Perhaps the following, derived from (f), may not be utterly unworthy of notice.

$$\left.\begin{array}{l} \frac{d^n\phi(x)}{dx^n} = \phi(D)\, o^n \varepsilon^{ox}, \quad \int^n \phi(x)\, dx^n = \phi(D)\, o^{-n}\, \varepsilon^{ox}, \\ \Delta^n\phi(x) = \phi(D)\, (\varepsilon^o - 1)^n\, \varepsilon^{ox}, \quad \Sigma^n\phi(x) = \phi(D)(\varepsilon^o - 1)^{-n}\, \varepsilon^{ox}, \end{array}\right\} \ldots (o),$$

where $\Delta x = 1$. We must not be startled at such quantities as 0^{-n}; for by means of the arbitraries of integration, all terms containing such quantities may be made to disappear.

In the development of such theorems as those which have been investigated, formulæ to facilitate reduction, similar to those which Sir John Herschel has given in his Examples, might be of use. But it would take up too much room to enter upon the subject here. We will only observe, that the theorems themselves will supply such formulæ, by giving particular values to x, particular forms to the function ϕ, and by comparing the coefficients of the same term, given by different developments of the function. And we may also change the function ϕ and the variable x, as we have done in this paper; and thus may multiply formulæ.

For example, by Taylor's theorem,

$$\varepsilon^{xD}\, o^n = x^n, \quad \varepsilon^{2xD}\, o^n = 2^n x^n, \quad \&c.$$

Also $\varepsilon^{D} o^n = 1^n, \quad \varepsilon^{2D} o^n = 2^n, \quad$ &c.

Therefore $\varepsilon^{xD} o^n = x^n \varepsilon^{D} o^n, \quad \varepsilon^{2xD} o^n = x^n \varepsilon^{2D} o^n, \quad$ &c.

And hence also $f(\varepsilon^{xD})\, o^n = x^n f(\varepsilon^{D})\, o^n.$

Or $f\{(1+\Delta)^x\}\, o^n = x^n f(1+\Delta)\, o^n.$

Or $f(xD)\, o^n = x^n f(D)\, o^n.$

If we expand (a) by the powers of lx, and if we change x in (l) into ε^{lx}; and expand that in like manner, and then compare like terms; we shall find

$$D^n\, \phi(\varepsilon^0) = \phi(1+\Delta)\, o^n.$$

We have treated only of the general form $\phi(x)$; but it is in certain particular functions, that the symbols of operation D and Δ give those simple expressions of them, which afford such easy and elegant means of integration. And here too we can sometimes employ other and more complex symbols than D and Δ with great effect.

Gunthwaite Hall, near Barnsley, Dec. 11, 1846.

ON PRINCIPAL AXES OF A BODY, THEIR MOMENTS OF INERTIA, AND DISTRIBUTION IN SPACE.

By Richard Townsend.

(*Continued from* p. 42.)

27. Any or all of the above constructions for principal axes (17, 18, 26) verify the anticipations of (2), shewing immediately, that an axis taken at random in a body may not be a principal axis at all; those axes alone being principal which are normals to surfaces of the second order confocal with the ellipsoid of gyration; and their principal points, or the points for which they are principal, being the points on these surfaces at which they are normals, and their corresponding principal planes being of course the tangent planes at those points to the same surfaces.

In distinguishing between axes in a given body, we have therefore to determine respecting every given axis, 1st, whether it be principal or not; and, 2nd, if it be, where will be its principal point, or points, if it have more than one.

Towards this we have the well-known theorem:

The normal and tangent plane at every point of any surface of the second order will meet each of its three principal planes in a point and line, which will always be pole and polar to each other with respect to the focal conic in that plane.

Hence, to find whether an assumed axis is principal or not, draw any plane perpendicular to that axis, and produce both axis and plane to meet a principal plane of the ellipsoid of gyration. If then the line of meeting of the plane be parallel to the polar of the point of meeting of the axis with respect to the focal conic in that principal plane, the axis will be principal, but otherwise it will not.

To find the principal point on the axis when the necessary condition is fulfilled, we have but to draw through that polar a plane perpendicular to the axis, that plane will meet it at its principal point, and for that point will be itself a principal plane.

(As the point on a principal axis for which it is principal has been called the principal point of that axis, so may the plane which at the principal point of a principal axis intersects it at right angles, as containing the other two accompanying principal axes, be called the principal plane of that axis.)

From the same theorem, it appears that in every plane drawn at will in a body, there exists a point for which one principal axis is perpendicular to that plane; that point in every plane may be called its principal point, since for it the plane, as containing two of its principal axes, is a principal plane.

To find that point in a given plane, from the pole of the right line in which it intersects a principal plane of the ellipsoid of gyration with respect to the focal conic in that plane, let a perpendicular be dropped on the given plane; that perpendicular will meet it at its principal point, and for that point will be itself a principal axis.

(As the plane perpendicular to a principal axis at its principal point may be called the principal plane of that axis, so may the axis perpendicular to a principal plane at its principal point be called the principal axis of that plane.)

Using for convenience these definitions, it appears at once from the above that, *every* plane has a corresponding principal axis, and that, every *principal* axis has a corresponding principal plane.

28. The above general test for principal axes, and general construction for their principal points, may be applied, and of course hold, in all particular cases: but there are some very important particular cases of principal axes which, being of frequent occurrence, should be familiarly known

without requiring the application of either; nor indeed is such application in their case necessary, as they appear more readily from the following simpler considerations.

If from any vertex situated on one of its axes a cone envelope any surface of the second order, then will one axis of that cone always coincide with that axis, and its remaining axes will be always parallel to the remaining axes of the surface.

Hence, in a body, a central principal axis is principal at every point along its whole length, and at all its points the other principal axes remain always parallel to each other and to the other central principal axes.

The three infinitely distant right lines in which the three principal planes of any cone of the second order, or of any hyperboloid of one or of two sheets, intersect the plane infinitely, will always be normals, both to that surface itself and also to its whole system of confocal surfaces, and will each be met by every different surface of that system obviously at a different point.

Hence, in a body, the three infinitely distant right lines in which the three central principal planes intersect the plane infinity, are also three other principal axes which are always principal at every point along their whole length.

The three central principal axes and these three particular infinitely distant axes are in this respect unique; they alone possess the property of being principal at *every* point, while all other principal axes are principal for but a single point.

Every line passing through the centre of a sphere is always a normal to that surface, and every system of surfaces of the second order confocal with an ellipsoid contains always a concentric sphere of infinite radius. Hence,

All axes passing through the centre of gravity of a body are principal axes, and the points for which they are principal are situated all at an infinite distance.

That system of principal axes alone possesses this property; all other principal axes which are not themselves infinitely distant having their principal points at a finite distance.

Moreover, since they all pierce the sphere of infinite radius perpendicularly at their principal points, it follows that the plane infinity is principal *at every point*.

Every line perpendicular to a principal plane of a surface of the second order is always a normal to the infinitely flat confocal surface, which is bounded by the focal conic in that plane. Hence, in a body,

At every point of a central principal plane one principal axis is always perpendicular to that plane, so that all axes parallel to a central principal axis are principal, and also, a central principal plane is principal *at every point.*

The three central principal planes and the plane infinity are in this respect unique; they alone possessing the property of being principal at every point, and every other plane having but a single point for which it is principal.

The poles of all diameters to a conic are always infinitely distant from its centre. Hence, in a body,

The principal points of all planes passing through the centre of gravity are situated all at an infinite distance.

That system of planes alone possesses this property; all other planes which are not themselves infinitely distant having their principal points at a finite distance.

Every line lying in the plane of a conic is a normal either to it or to some confocal conic; and to find the point for which it is a normal, drop upon it a perpendicular from its pole with respect to the given conic, the line and perpendicular will then be the normals to the two confocal conics which pass through their points of intersection. Hence, in a body,

All axes which lie in a central principal plane are principal axes; and to find the points for which they are principal, from the pole of each individual axis with respect to the focal conic in that plane of the ellipsoid of gyration, let a perpendicular be dropped on that axis; the point of meeting will then be the principal point, and the perpendicular will be the accompanying principal axis.

The same is true of the plane infinity, all axes which lie in that plane being also principal; for, the asymptotic cones to the system of hyperboloids confocal with the ellipsoid of gyration intersect at right angles the plane infinity in a system of confocal conics; and to some one of these, and therefore to the particular surface on which it lies, every line taken at will in that plane is a normal at some point or other, and therefore a principal axis of the body.

The common foci of this system of conics are the points where the assymptotes of the focal hyperbola to the ellipsoid of gyration pierce the plane infinity. Hence at once a construction for finding the principal point of an infinitely distant axis, the same exactly as determines it for an axis lying in a central principal plane; and hence also the construction established on other principles in Art (26) for finding the principal axes at an infinitely distant point.

Join that point with the centre of gravity, the joining line will be one principal axis; and to find the other two, draw two planes passing through that line and the asymptotes of the focal hyperbola, and bisect the two supplemental angles, acute and obtuse, between the lines in which they meet the plane infinity, the bisecting lines will be then the two principal axes sought.

At every point in a central principal plane the three principal axes may also, and for the same reason (26), be found by a construction quite elementary; at the point erect a perpendicular to the plane, it will be one principal axis, and to find the other two, connect the point with the two foci of the focal conic in that plane of the ellipsoid of gyration, and bisect the supplemental angles, acute and obtuse, between the connecting lines, the two bisectors will be then the remaining principal axes sought.

The three central principal planes and the plane infinity alone possess the above property; for no other plane are *all* axes which lie therein principal, though (as shall presently appear) there exists in every plane *an infinite number* of principal axes which are always distributed therein according to a simple and very elegant law.

The principal results contained in this article may be briefly summed up, by saying, that in every body the three central principal planes and the plane infinity possess always the remarkable and unique properties—that every axis perpendicular to, and also every axis lying in, any one of these four planes is always principal; that they are always themselves principal at *every* point; that all axes passing through any one of their four points of intersection are always principal; and that their six lines of intersection, all principal axes, possess always the important and exclusive property of being principal for *every point* along their whole length.

29. The general test (27) for ascertaining whether an axis be ordinary or principal leads at once to the following, which often serves the same purpose more easily, and also leads itself to some very general geometrical properties of principal axes.

The right line in which any two planes taken at random in a body intersect will be a principal axis if the principal axes of those planes intersect, but otherwise it will not; and when they do intersect, their plane will be the principal plane corresponding to that principal axis.

For producing the planes with their principal axes to meet one of the principal planes of the ellipsoid of gyration, the points of meeting of the axes will be the poles with respect to the focal conic in that plane of the traces of the planes, and therefore the intersecting point of these traces will be the pole of the line joining the feet of the axes: if then the axes intersect, a plane passing through this line and containing them both, will be perpendicular to the intersection of the planes, which intersection will therefore (27) be a principal axis at the point of meeting; but if the axes do not intersect, no plane passing through that line can be parallel to them both, that is, perpendicular to the intersection of the planes; in that case, therefore, that intersection (27) will not be a principal axis.

Conversely, (for the same reason) if any two planes be drawn at will through a principal axis, the corresponding principal axes will intersect, and their plane will be the principal plane of the axis; but of no two planes drawn through an axis which is not principal, will the principal axes ever intersect.

Now, through an axis, of which we do not know whether it is principal or not, we may often be able to draw two planes so conveniently as to be able to determine immediately their principal axes: if we can do this, the nature of the axis will by means of the above be immediately determined. Instances will appear as we proceed.

Hence, also, we immediately deduce the following consequences:—

If a system of planes pass all through a *principal* axis, their corresponding system of principal axes will lie *all* in the same plane, viz. the principal plane of the axis.

But if a system of planes pass all through an axis which is not principal, then will no two of their whole corresponding system of principal axes lie in the same plane.

We shall presently see that in the latter and obviously more general case, the system of axes will always generate an hyperbolic paraboloid, and that in the former particular case they will always envelope a parabola; in fact, the *gauche* surface, which alone in the general case could be generated, degenerates in the particular case into a developable, and the limit to an hyperbolic paraboloid, when it flattens down into a developable surface, is obviously a portion of a plane bounded by a parabola, the generatrices of that limiting surface being the tangents to that curve.

More generally, if in place of passing all through the same right line, that is, touching all an infinitely slender evanescent cylinder, a system of planes be tangents all to any developable surface, then will the surface which their corresponding system of principal axes obviously generates be also developable, if all the edges of the surface touched be principal axes; but if they be not principal axes, that surface will be gauche.

For in one case every successive pair of planes intersect in a principal axis, and in the other they do not: in the former case, therefore, every consecutive pair of generating principal axes will intersect, and in the latter they will not.

And conversely, if a system of principal axes be so restrained by some governing law as to generate a developable surface, then will the surface enveloped by their corresponding system of principal planes be obviously another developable, all whose edges will be also principal axes; but if the surface generated by the system of principal axes be gauche, then will no edge of the developable envelope of the corresponding system of principal planes be a principal axis.

Hence, for every continuous curve which could be traced out in a body, plane or of double curvature, such that all its tangents would be principal axes, there exists a second curve intimately connected with the first and possessing the same property, the arête de rebroussement, viz. of the developable surface envelope of the system of principal planes corresponding to that system of tangents; these two curves are moreover reciprocally convertible with each other, the first also being obviously the arête de rebroussement of the developable envelope of the system of principal planes corresponding to the tangents of the second.

In every body there exists an infinite number of pairs of curves thus reciprocally connected: of these we shall meet with instances as we proceed, and we shall also see that they are always connected by *another* and different relation of reciprocality, by means of which all the properties of either may be immediately deduced from those of the other as reciprocally correlative.

If, as often happens, one of these curves be such that all its tangents are not only principal axes, but moreover principal at their points of contact, then will the corresponding system of principal planes be obviously the system of normal planes to that curve, and the other curve being the arête de rebroussement of the developable envelope of that

system of normal planes, will be connected with it by the well-known relations which exist between two such curves. And, again, if one of the two developables be such that all its tangent planes have their principal points on their edges of contact, then obviously will the other developable intersect it along the curve locus of these points, which curve, as all the edges of the second are in that case normals to the first, will be a common involute to their arêtes de rebroussement, that is, will be a line of curvature common to both.

And if in the latter case that line of curvature be made the arête de rebroussement of a third developable surface, then will all the edges of that developable, that is all the tangents to the curve, be also principal axes; and moreover principal at their points of contact, for at each point of that curve the normal plane contains the edge of the first developable, which is a principal axis, and also the normal to that surface which, by hypothesis, is also a principal axis, principal at that point, and therefore the tangent to the curve is the third principal axis, principal at its point of contact. Hence, moreover, the system of normal planes to that curve, which pass through the edges of the first developable and are at right angles to its tangent planes, will envelope a fourth developable, all whose edges will be also principal axes.

A system of four developable surfaces thus connected are of not unfrequent occurrence: we shall just delay to give a single example illustrative of the principles contained in this article.

30. Let a developable surface circumscribe the ellipsoid of gyration or any one of its system of confocal surfaces, then will all its edges be principal axes, if the curve of contact be a line of curvature; but otherwise they will not.

For the system of principal axes corresponding to the enveloping tangent planes are in this case the normals at their points of contact, the edges therefore of the developable envelope will be principal axes, if the normals intersect two and two consecutively; but otherwise they will not.

The converse of this we might easily see, *à priori*, from the known property that every two of the confocal system of surfaces intersect at right angles in a line of curvature common to both, and therefore the edges of the developable which circumscribes either along that intersecting curve are a system of normals to the other, and therefore a system of principal axes.

In the present case also the principal point of every tangent plane to either of the two developables is on its edge of contact, hence (29) the two surfaces intersect in a line of curvature common to both; which is also evident *à priori*, since, if any surface whatsoever envelope another along a line of curvature, the curve of contact will be obviously a line of curvature also of the enveloping surface.

Hence, also, (29) if that curve be made the arête de rebroussement of a third developable surface, all its edges, that is all the tangents to the curve, will be principal axes, principal at their points of contact, which in the present case is also evident *à priori*, for the tangent at every point on the curve of intersection of any two of the surfaces confocal with the ellipsoid of gyration is the normal to the third confocal surface which passes through that point; the whole system of tangents all round that curve is therefore a system of principal axes, principal at their points of contact.

The developable surface generated by the system of tangents all round a line of curvature of any one of the surfaces confocal with the ellipsoid of gyration, we shall have occasion to notice again, for all its edges are not only principal but also equimomental axes, and the whole system of such, subdivisible in various ways into an infinite number of smaller systems according to different arbitrary laws of division, possesses many curious and interesting properties.

The fourth developable (29) remains still to be noticed: if, therefore, we take any line of curvature on any one of the whole system of surfaces confocal with the ellipsoid of gyration, the system of normal planes to that curve will generate by their successive intersections a developable surface, all whose edges will be principal axes.

31. *Every* ruled surface, whether gauche or developable, which is generated by a system of principal axes, is connected with the corresponding developable surface envelope of the corresponding system of principal planes by the following relation; from which if either be given, the other may be readily determined.

Their curves of intersection with each principal plane of the ellipsoid of gyration are always polars reciprocal to each other with respect to the focal conic in that plane.

For, the points in which the different principal axes pierce that plane are the poles with respect to the focal conic therein of the lines in which the corresponding principal planes intersect the same; the curve, therefore, envelope

of the latter is the polar reciprocal with respect to that conic of the curve locus of the former.

This property, which is but a particular case of one more general, will be often found very useful, for in several remarkable cases one of these curves in each principal plane is easily seen to be a conic, and in all such cases we see from this that the other also must be of the second order.

Suppose, for instance, that a system of planes generate by their successive intersections a developable surface circumscribing the ellipsoid of gyration, or any one of its confocal system, along its curve of intersection with any concentric and coaxal surface of the second order, or more generally along any curve whose orthographic projections on the three central principal planes are conics, then will the surface generated by their corresponding system of principal axes intersect the same three planes also in conics; for in that case, the curves of intersection of such a developable being polars reciprocal with respect to the principal sections of the surface enveloped to the orthographic projections of the curve of contact on the same respectively, are of the second order, and therefore so are also *their* polars reciprocal with respect to the focal conics in the same planes.

Hence, again, if (as often happens) a system of planes determined by some law envelope a cone of the second order, as, for instance, if such a cone be taken arbitrarily in the body, and that the system of planes be its system of tangents, then will the surface generated by the corresponding system of principal axes intersect the three central principal planes in conics; these three conics will be all parabolas, if the cone pass through the centre of gravity; and if its vertex lie in either of the three central principal planes, the conic in that plane will dwindle into a finite portion of a right line, for the particular conic in which the cone intersects that plane will be in this case two right lines real or imaginary, and the polar reciprocal with respect to any curve of the second order of such a conic is always infinitely flat, the portion viz. of a right line bounded by the poles of the two lines; and if moreover the cone touch that plane, then will the corresponding conic dwindle into a point, the pole with respect to the focal conic of the side of contact.

And, conversely, if a system of principal axes generate a surface of the second order, then will the developable surface envelope of the corresponding system of principal planes intersect each central principal plane in a conic;

these three conics, as above, will be all parabolas if the generated surface pass through the centre of gravity. And if (as not unfrequently happens in consequence of every axis which lies in either of the three planes being (28) principal) it touch any of the central principal planes, that is if it intersect it in two right lines, then apparently will the conic in which the corresponding developable intersects that plane be infinitely flat, the right line viz. which passes through the poles of the two lines with respect to the focal conic: in this case, however, since one of the two lines must be one of the generating axes, the whole right line is due to *its* principal plane, and the conic is therefore properly but a point, the pole of the other line.

If the surface of the second order generated by the system of principal axes touch the three central principal planes, then will the three conics in which the developable envelope of the corresponding system of principal planes intersects those planes, all dwindle into points; that developable therefore in this case will be an infinitely slender cylinder. Hence we see that if a surface of the second order generated by a system of principal axes touch any two of the central principal planes, it must also touch the third, and that if a system of planes pass all through the same right line, that is, if they all touch an infinitely slender cylinder of the second order, then will the surface generated by the corresponding system of principal axes touch the three central principal planes; that surface will in fact be a paraboloid of the second order. But of this more hereafter.

In general, if the developable surface envelope of any system of planes be an infinitely flat cone or cylinder of any order, and therefore intersect the central principal planes in finite portions of a right line bounded by points, then will the surface generated by the corresponding system of principal axes touch those three planes, for it intersects them in right lines, the poles of the above points, and every plane which passes through a right line on a gauche surface is a tangent plane to that surface at some point or other.

32. As an example illustrative of the preceding article, Let a system of planes touch the ellipsoid of gyration or any one of its confocal surfaces along any plane section; their developable envelope will be of course a cone of the second order, intersecting the three central principal planes in conics, and the corresponding system of principal axes, that is the system of normals to the surface along the plane curve,

will generate a gauche surface of the fourth order, which (31) will therefore intersect the same planes also in conics, the remaining portion of each curve of the fourth order consisting of two right lines real or imaginary, the particular pair of generating principal axes normals to the surface of the second order at the two points, real or imaginary, where the plane of the curve of contact intersects each principal section of that surface.

Let the plane of the section pass through the centre of gravity; then will the cone become a cylinder, and its intersection with each central principal plane a conic, concentric with the principal section on that plane. Hence we see that every system of principal axes normals to any one of the surfaces confocal with the ellipsoid of gyration along any central section of that surface, will intersect each central principal plane in a conic whose centre will be the centre of gravity,* the remaining portion of each curve of the fourth order, in which the surface generated by that system of axes intersects these planes, being two real and parallel right lines, the pairs of normals viz. to the surface of the second order at the diametrically opposite pairs of points, in which the plane of the central section intersects each central principal section.

Let now the particular surface be one of the confocal hyperboloids of one sheet, and let the plane of the central section be one of the system of tangent planes to its asymptotic cone. In this case the section will be two right lines parallel to each other and to the side of contact; and since all planes passing through either of these are tangent planes to the surface, the cylinder enveloping it along that section

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* The same principle enables us to find immediately the curve in which the surface locus of its centres of curvature intersects each principal plane of any surface of the second order: that curve consists of two parts, corresponding to the two sheets of the surface of centres; of these, one in each plane is obviously the evolute of the principal section in that plane, and to find the other we have but to find the locus of the ultimate intersections with that plane of a system of normals drawn infinitely near to it, which is immediately done by means of the principle in question; for the envelope of the intersections with the same plane of the corresponding system of tangent planes to the surface is ultimately the principal section therein, and of this envelope the locus required is the polar reciprocal with respect to the focal conic in that central plane. Hence in every surface of the second order, the curves of intersection with each principal plane of its surface of centres are the evolute of the principal section in that plane, and, a conic concentric and coaxal with, and having its semiaxes, thirds proportional to those of the focal and principal conics in the same principal plane.

will be infinitely flat, being either portion indifferently of the central plane divided into two regions bounded by the two parallel lines. Such being always the limiting and transition state between an elliptic and an hyperbolic cylinder, and the system of tangent planes to its asymptotic cone determining on every hyperboloid of one sheet the bounding system of central sections, on one side of which the enveloping cylinders to that surface are all elliptic, and on the other side of which they are all hyperbolic, the particular class of parabolic enveloping cylinders being confined to the two paraboloids.

Since, here, the system of planes touching the hyperboloid pass all through two parallel right lines, equidistant from and in a plane passing through the centre of gravity, it follows that the developable surface, their envelope, intersects each central principal plane in two points equidistant and in opposite directions from that centre, and therefore the gauche surface generated by the corresponding system of principal axes will (31) touch these three planes, and besides intersecting them each in the two normals to the hyperboloid at the two diametrically opposite points in that principal plane, will also intersect them in two parallel right lines, the polars of these points with respect to the focal conic.

The complete curve of the fourth order in which the surface generated by the normals intersects each central principal plane, consists therefore in this case of four real right lines, in pairs parallel to each other and similarly situated in opposite directions from the centre of gravity, one pair of these parallel lines being generating axes, and the other two forming the polar reciprocal with respect to the focal conic of the particular curve, in which the developable envelope of the corresponding system of principal planes intersects that central plane; a result confirmatory of the concluding remarks in (31).

That such should be the nature of the intersecting curves we might easily have seen *à priori*, for in the particular case in question the gauche surface of the fourth order, generated by the system of principal axes, breaks up into two paraboloids of the second order, equal, similar, and similarly placed, but in opposite directions from the centre of gravity, one corresponding to one of the two parallel generatrices of the hyperboloid of one sheet, and the other to the second; they both touch the three central principal planes, and therefore intersect them each in two right lines, the analogous lines for each being of course parallel to each other.

For, taking arbitrarily any rectilinear generatrix of any one of the confocal hyperboloids of one sheet, the system of principal axes normals along it to the surface will generate an hyperbolic paraboloid, for (27) they pass all through the three polars with respect to the focal conics of the points, whence the assumed generatrix pierces the three central principal planes, and therefore generate a surface of the second order. Again, they are all perpendicular to that generatrix, that is, all parallel to the same plane, and therefore the generated surface is a paraboloid.

This paraboloid intersecting each central principal plane in a right line, of course intersects it in another, and therefore touches it; that other is the particular generating principal axis which lies in the principal plane, that is, the normal to the principal section of the hyperboloid in that plane at the point where the generatrix meets it; the intersection therefore of that normal with the polar of the same point with respect to the focal conic, is the point of contact of the paraboloid. Also, since one of its lines of intersection with each plane is one of the generating principal axes of that surface, the corresponding system of principal planes will intersect that plane in a system of lines whose envelope ought (31) to be an evanescent conic, the pole of the other line of intersection with respect to the focal conic; and such, since they all pass through a line, it is in the present instance. But we forbear at present to consider this surface any further, as it will be fully and more generally discussed in a subsequent article.

33. Having spoken of curves plane or of double curvature enveloped by principal axes, and of rule surfaces gauche or developable generated by systems of principal axes, it may, before we proceed, be satisfactory to examine briefly the possibility of either taking place, and the conditions which are necessary to be fulfilled in such cases.

Now a principal axis being a normal to one of a system of surfaces which (VIII.) contain in their equation a variable parameter δ, its equations contain in their parameters four variable quantities, viz. xyz, and δ where xyz are the coordinates of the point on the surface δ at which it is a normal; and these four quantities are connected by but a single equation, that of the surface, so that of the four parameters in the equations of a principal axis three are absolutely independent.

In order therefore that a system of principal axes should generate a ruled surface of either species, it is necessary that

they be restricted by two conditions; and for the same reason, conversely, if a system of principal axes be restricted to fulfil any two independent conditions, the axes of that system will generate a surface gauche or developable as the case may be.

In the latter case they will obviously be all tangents to a curve, the arête de rebroussement of the generated surface; in order therefore that a system of principal axes should envelope a curve, it is necessary that the axes of the system be restricted by two conditions: but, conversely, it does not always follow that a system so restricted will envelope a curve, unless that the given conditions be such as to involve in their very nature the additional circumstance that the surface generated by the system of axes conformable to them must be developable. But in general this is a result purely accidental, and arising in particular cases from some coincidence between the conditions of a nature altogether casual; since, when the principal axes of a system are restrained by two arbitrary and independent conditions, the surface they generate is of course completely fixed, and may or may not be developable as the case may be.

Suppose, for instance, that a system of principal axes be restrained to lie all in a given plane; here being restricted by two conditions, they will envelope a curve in that plane, the arête de rebroussement of the developable surface, which from the very nature of the conditions is in this case necessarily described. Or suppose that they be all required to pass through a given point, here also being restrained by two conditions, they will generate a surface, which from the nature of the conditions is in this case also developable. But let the conditions be, that they all pass through two given curves, plane or of double curvature, or that they all touch two given surfaces, or that they all touch a given surface along a given curve, then is it plain that the fact of the resulting surface being developable would be purely accidental, and arise from a coincidence peculiar to the particular case proposed.

If one of the two conditions be that every consecutive pair of the axes should lie in the same plane, and therefore intersect, then certainly would the resulting surface generated by the system of axes be necessarily developable, whatever might be the other condition: but then the question would be indeterminate, and a given single condition combined with the first could only determine the nature of the developable, but never in any case the particular surface itself. For every system of principal axes which are restricted to

fulfil but a single condition may always be divided into a multitude of smaller systems, each forming a developable surface, and consequently the problem, to find the developable surface generated by a system of principal axes which fulfil one given condition, is essentially indeterminate.

Suppose, for instance, it were required to find the developable of principal axes which should pass through a given curve, the problem would be indeterminate: for, from each point of the curve there diverges a cone of principal axes, and we might select at random a side of one of these cones as an edge of our developable; then take the infinitely near side of the consecutive cone which intersects that as the second edge, and so on, and thus get an indefinite number of developable surfaces, three being an indefinite number of sides to a cone. The cone, which from the centre of gravity as vertex passes through the given curve, fulfils the conditions (since all axes passing through the centre of gravity are (28) principal), and is therefore one of these developables; and so are also the three cylinders which orthographically project the curve on the central principal planes, since all axes perpendicular to any of those planes are principal (28). The same exactly might be said if it were required to circumscribe a given surface with a developable of principal axes. Or suppose that it were required to find a developable of axes in a body which should be all both principal and equimomental, then (30) should we have a multitude of developables fulfilling the conditions, of which the arêtes de rebroussement would be lines of curvature on the sufaces of the system confocal with the ellipsoid of gyration. In this last instance the developables present themselves to our consideration immediately and naturally, and their arêtes de rebroussement generate, by the continuity of their change of position from surface to surface, a very remarkable and familiar surface, which will form the subject of a subsequent article. (See Art. 15, p. 25).

34. Let us now see what takes place when we have a system of principal axes restricted by but a single condition, or, algebraically speaking, whose parameters are connected by but a single equation.

In such a case, though they will envelope, they will not of course generate a surface, but the whole system comprising them may be divided in a multitude of different ways into an infinite number of smaller systems, each of which will form a continuous surface. For, we may arbi-

trarily *introduce* an additional and independent condition, or, which is the same thing, we may connect the parameters by a second and perfectly arbitrary equation, and thus detach from the system a group which fulfilling two conditions will generate a surface. We may then, in the introduced equation, cause an arbitrary constant to vary, and thus obtain a different group forming a new surface of the same species with the former; and finally, we may give to that constant all possible values in continued succession, and thus have the whole original system of axes divided into a multitude of groups forming a system of surfaces all of the same species. Again, this division may obviously be performed in an infinite number of different ways, for the number of equations containing each an arbitrary constant, which might be introduced between the parameters, is of course infinite.

Examples of this division of systems subject to but a single restriction will appear as we proceed, and also instances illustrative of the advantages which may be derived from the power we have of, 1st, Selecting in most cases whatever law of division we may find convenient; 2nd, Of sometimes changing that law when occasion may seem to require it; and, 3rd, Of considering the same given system as made up of two or more different and distinct groups of smaller systems according to different and arbitrary laws of division.

In every case of the division of a system of principal axes restricted by but one condition, the surfaces formed by the smaller systems, as containing each but a single parameter variable from one to the other, will admit of an envelope, this will obviously be the surface generated by their successive characteristics or the curves in which they intersect, two and two consecutively; and to find the equation of that surface, we have but to proceed in the usual manner, setting out from the equation containing but one parameter which expresses the system of surfaces enveloped.

Now, though there exists an infinite number of ways in which the division of a system of surfaces may be performed, and therefore an infinite number of groups of surfaces enveloped, still for all of them the envelope will be the same, but the circumstances of its determination will be considerably different in the different cases; these will readily appear from the following considerations.

Whenever we have a system of right lines which are restrained by any two independent conditions, or, which is the same thing, when their parameters are connected by two independent equations, that system will of course envelope

a surface, which surface is fixed and implicitly determined when the conditions are given. Hence, a system of principal axes which are restricted by but a single condition will always envelope a surface, which will, for the same reason, be fixed and implicitly determined when the condition is given.

This envelope, moreover, is obviously the same as that of any system of surfaces whatsoever into which we may divide the system of axes; hence, by whatever arbitrary law we may divide a given system of principal axes restricted by but one condition into an infinite number of smaller systems, each forming a surface, the resulting system of surfaces will have invariably the same envelope, the surface viz. which touches the whole given system of axes.

To find that surface when the system of axes is given, we have therefore but to introduce a condition, and having thus divided the system (as stated above) into an infinite number of surfaces, proceed from the equation, which containing one variable parameter expresses that system of surfaces, to determine their envelope in the usual manner.

But for every different introduced condition we have a different equation, and it is obvious that on the form of that equation depends the facility, and perhaps the possibility, of determining the equation of the envelope; we must therefore endeavour to find among all the different systems of surfaces into which the original system of axes is divisible, that particular system which has the simplest equation.

This would no doubt be often difficult, and no rule can perhaps be given which will hold in all cases. But, if we may conjecture any thing from the known general laws of envelopes, the most manageable form of the equation expressing the system of surfaces enveloped will in most instances correspond to the case where one of these surfaces, and therefore their whole system, is developable.

To find, therefore, the surface enveloped by a given system of principal axes, restrained by but a single condition, we must first divide the whole system into an infinite number of groups, each forming a developable surface, and then, having thrown into its simplest form the equation expressing that system of developables, proceed in the usual way to determine their envelope.

This particular way of performing the division of the system of axes possesses moreover the important advantage of enabling us (as we shall just now see) to form a tolerably

clear conception of the nature of the envelope itself, and at the same time it also leads indirectly to the result, that whenever there exists one way of forming from the axes a system of developable surfaces, then will there always exist a second and entirely different way of performing a similar division; and that, hence, generally there always exist two, and not more than two, different and distinct systems of developable surfaces, into either of which a given system of principal axes restricted by a single condition may be always divided.

Now, when the nature of the condition is such that the axes admit at all of a real envelope, the division of the system into at least one system of developable surfaces is always possible. For in all such cases, introducing at random any other condition whatever, we shall then, by the variation of the constant in the equation expressing the introduced condition, have the whole system divided into a series of rule surfaces of some sort or other, generally not developable; these surfaces, like every other system of consecutive surfaces which admit of a real envelope, will intersect two and two consecutively in a system of real curves, and through every point on the curve, in which any one of them taken arbitrarily from the whole system is intersected by the consecutive surface, there will pass a generatrix of each of these two surfaces, that is, a consecutive pair of the original system of axes will there intersect each other. Again, of these two generatrices at each point of this curve, one will always meet the consecutive curve which lies on its own surface, and through the point of meeting there will pass also a generatrix of the third consecutive surface, that is, a third consecutive axis of the given system will there meet the second. This will again meet the third consecutive curve, and through the point of meeting there will pass a fourth consecutive axis of the given system; and so on, at curve after curve, the same thing will take place successively, until the whole series of curves will be all exhausted. Hence, passing through a point on every curve of the whole system, we shall have a developable surface formed of a system of the original axes, and hence therefore, in the transition from point to point of any one individual curve, we shall have that whole system of axes completely exhausted, and divided into a series of developable surfaces.

The suggested division of every such system of principal axes into a system of developable surfaces, as a preparatory step towards endeavouring to find the surface envelope

of that system of axes, being therefore possible whenever that for which we seek has a real existence, we may suppose that division as having actually taken place in every individual instance, and we thus learn respecting the nature of the envelope in general. That, like the surface envelope of the whole system of normals to every algebraic surface, it consists always of two different and distinct sheets, which, like that same class of surfaces, may in some cases divide into two different and distinct surfaces, and which in others may dwindle either wholly or in part into a curve or evanescent surface, but which rarely, if ever, run into each other like the two sheets in Fresnel's "biaxal wave surface of double refracting media," or like the three sheets in Mr. Haughton's "surface of wave slowness of crystalline elastic solids," being for the most part separated from each other, and to the eye appearing to be two distinct surfaces, even in the great majority of cases, where, algebraically speaking, they are but parts of a single surface, and are contained, one and both, in the same unresolvable equation. To see this, let us take some one of the developable surfaces into which we may consider any particular system of axes divided, and let us follow it in its variation until the whole system be exhausted; we shall then perceive the envelope to consist always of a sheet generated by the system of consecutive curves, in which the different consecutive pairs of developables ultimately intersect, a sheet to which these developables will obviously be all circumscribed along their respective curves of ultimate intersection, and also of another and wholly distinct sheet generated by the system of arêtes de rebroussement of all these developable surfaces; a sheet to which, as well as to the other, the axes will be all tangents, but which with respect to *the same* system of developables will not be circumscribed by that system of surfaces, but will be merely the locus of their arêtes de rebroussement.

With respect to the enveloping system of axes, these two sheets possess however exactly the same properties. For, taking that second sheet with its whole system of lines of regression, that is, with its system of generating curves, considered in the above method, and conceiving as traced out upon it what may be called the conjugate system of curves, those viz. which all intersect every one of the former, so that the tangents to the two intersecting elements shall at every point of the surface be there a pair of conjugate tangents, let a system of developable surfaces be

circumscribed to the sheet along this new system of curves. Then, since every edge of a developable circumscribed to any algebraic surface forms always with the corresponding tangent to the curve of contact a pair of conjugate tangents to the surface at its point of contact, will that system of circumscribing developables possess the property that their edges will be all tangents to the original system of curves, and therefore also all tangents to the other sheet of the envelope and consequently all principal axes of the original system. This new system of developable surfaces, different altogether and distinct from the system which we have considered as producing the envelope, is therefore formed also out of the original system of principal axes, and equally with the former exhausts that whole system. Again, the system of curves arêtes de rebroussement of the new developable system must lie all upon the first sheet of the envelope; for, the edges of each surface of that system being all tangents to that sheet, if they did not, the developables themselves would be all circumscribed to the first sheet also, and therefore to the two sheets together: this certainly might take place for one particular surface of the system, which surface in certain cases might even break up into two, three, or more different surfaces; but it would be impossible that the whole system infinite in number should be all circumscribed to the two sheets simultaneously. Hence always the arêtes de rebroussement of the new system lie all on the first sheet, upon which, for the same reason as above, they are, conversely, the system of curves conjugate to the lines of contact of the former system.

Hence we see that, for every system of principal axes restricted by a single condition, there exist always two and but two different and distinct systems of developable surfaces, into either of which that whole system may be always divided; and also that the surface envelope of every such system of axes consists generally of two different and distinct sheets, separated from, and rarely if ever running into each other, of which sheets each will be enveloped by one of the two component systems of developable surfaces into which that system of axes may be divided, and will be the locus of the arêtes de rebroussement of the other system, and upon both of which the two opposite systems of generating curves, the lines of contact and the lines of regression, will be always conjugate to each other. These properties bear an obvious and close analogy to those of the whole system of normals to every algebraic surface, for

every such system of right lines in space, there being always two different and distinct systems of developable surfaces, into either of which they may be always resolved, and the surface their envelope being always of the same nature as that just considered in the present case.

As in the latter class of surfaces, it is obvious that if, in the class of envelopes we are now considering, either of the two systems of lines of regression be all plane curves, then will the sheet generated by the other system be always a developable surface; for that sheet, being the envelope of the system of developable surfaces of which the plane curves are the arêtes de rebroussement, will in that case be the envelope of a system of planes whose common equation contains but a single variable parameter. It will be presently proved (but let that be assumed, if necessary, throughout the present article) that every *plane* curve, whose tangents are all principal axes, will be always a parabola of the second order.

Such is the general type of the surface envelope of a system of principal axes restricted by a single condition: like the class of surfaces to which we have compared it, it is generally (algebraically speaking) a single surface consisting of two distinct sheets, which for the most part are separated from and rarely if ever run into each other, though as in their case the two sheets may sometimes be two separate surfaces, or even one of them or perhaps both may dwindle into a curve; but respecting the individual sheets themselves, whatever be their nature, there is nothing whatever which similarly restricts their character or limits them to any particular class, description, or form of surfaces, they may each or both, as the cases may be, have themselves one, two, or any number of sheets, they may either or even both be developable surfaces, they may be both closed or both open, or they may either or both be limited in one direction and extend to infinity in the other, or either or both may return back into themselves or extend to infinity in any or in every direction; their curve or curves of intersection moreover may be of any nature whatever, they may be altogether imaginary or they may be wholly or partly real, and if real wholly or in part, they may wholly or partly be closed and return back into themselves, or they may wholly or in part be open and extend to infinity in any or in every direction.

Suppose that a system of principal axes were restricted by the single condition of passing all through a curve

plane or of double curvature assumed arbitrarily in the body; then, from every point of that curve will a cone of the axes diverge, this system of cones will be one of the two component systems of developable surfaces into which that system of axes may be divided. The other also may be easily found: take arbitrarily a side of any one of these cones as the basis of a developable of the new system, this will intersect the consecutive cone in a number of points equal to the order of that cone; of the sides of that cone passing through these points take that which is consecutive to the assumed side of the first, this will be the second side of the developable, and will intersect the cone from the third consecutive point; of this the consecutive intersecting side will be the third edge of the developable, and so on. The developable so found will be one of the second system, and the others of that system may be found in a similar manner. In this case the two sheets of the envelope will be quite distinct from each other; one, the locus of the arêtes de rebroussement of first system of developables, that is, the locus of the vertices of the system of cones will be the assumed curve, the other will generally be a surface, the locus of the arêtes de rebroussement of the other system of developables, or, which is the same thing, the envelope of the system of cones.

Hence we see that in every body there exists an infinite number of systems of principal axes restricted by a single condition, for which one of the sheets of the envelope will not be a surface, but a curve plane or of double curvature as the case may be.

Suppose again that a system of principal axes were restricted by the single condition of being tangents all to a developable surface given or arbitrarily assumed in the body. Here again as in the former example we readily obtain the two component developable systems and the two sheets of the complete envelope; for in every tangent plane to the assumed surface there lies an infinite number of principal axes which envelope a parabola in that plane; hence the system of tangent planes themselves to that surface, or rather the system of portions of each tangent plane bounded by their respective parabolas, will be one of the two component developable systems: the other, as in the example above, may also be easily found, take arbitrarily a tangent to the parabola in any one of the tangent planes, as the basis of a developable of the new system, this will intersect the consecutive tangent plane in a point, of the

two tangents to the parabola in this new plane which pass through the point of meeting take that which is consecutive to the first assumed tangent, this will be the second edge of the developable and will meet the third consecutive tangent plane in a point, the consecutive tangent through which to the parabola in that third plane will be the third edge of the developable, and so on to the end. The developable so found will be one, the second system and the others of that system may be found in a similar manner; in this case also the two sheets of the envelope will be quite distinct from each other, one, the envelope of the first system of developables, that is of the system of tangent planes to the assumed developable surface, will be that developable surface itself, the other, the locus of their system of arêtes de rebroussement, will be the surface generated by the system of parabolas envelopes of the systems of principal axes in the system of tangent planes to that same surface.

Hence we see that in every body there exists an infinite number of systems of principal axes restricted by a single condition, for which one of the sheets of the envelope will be a developable surface; also, that in all such cases, one of the two component developable systems will be always a system of planes, the system of tangent planes to the developable sheet itself; and that moreover, the system of lines of contact with that sheet will be always a system of right lines, the system of edges of the sheet itself, while the system of lines of regression on the other sheet will be always a system of plane curves and all parabolas of the second order. As for the second sheet itself, the system of lines of contact on that sheet, and the system of lines of regression on the developable sheet, they obviously all vary in their nature and properties with the system of axes themselves, and remain to be determined when that system is given in every particular case.

Again, more generally, suppose that a system of principal axes were restricted by the single condition of being tangents all to a surface of any nature whatever, given, or arbitrarily described in the body, and that it were required to find the two systems of component developables, and the surface envelope of the system of axes. Here, as indeed also in the preceding case, we might at first sight suppose that the latter were already known and that the surface itself were the envelope, but this would not be the case exactly; the surface itself, unless indeed (which of course would scarcely

ever happen) it chanced to be given or assumed so fortunately as that every tangent which was a principal axis would have double contact with it, would not be the whole envelope, it would be only one of its two sheets, and the other, which therefore in the vast multitude of cases of this nature will be always a distinct surface, would still remain to be determined.

Hence we see that in every body cases without number exist, for which the two sheets of the surface envelope of a system of principal axes restricted by a single condition are two distinct surfaces; and, moreover, that these surfaces are not confined to any particular class or species, but though of course in every case inseparably connected with each other, that either may be of any nature whatever; for in that extensive class of cases where a system of principal axes are restricted to touch a given surface, that surface itself which is absolutely arbitrary is always one of the sheets of the envelope, and the other fixed of course and implicitly determined when the first is given, may, according to the varieties of that first, be also of any nature whatever; the curve or curves, moreover, in which the two surfaces intersect each other, may, as depending on these surfaces, be also of any kind whatever, it may be altogether imaginary or it may be real, and if real, it may either be one continuous curve closed or extending to infinity, or it may consist of two or more detached curves separated from each other by intervals of both surfaces. If the two surfaces happen to touch each other at one or more points, then obviously will every point of contact be a double point, nodal, cuspal, or conjugate, on the curve of their mutual intersection.

It is not to be supposed however that every point of these two surfaces, or more generally of the surface whatever be its nature envelope of a system of principal axes restricted by a single condition, is in all cases actually touched by an axis of the enveloping system; on the contrary it more frequently happens that only a portion of the whole envelope is actually available, and that upon that surface, or to speak more generally, upon every surface whatever given or arbitrarily assumed in a body there exists two distinct regions, continuous or discontinuous, separated from each other by a very remarkable curve or curves, such that for all the points of one region some among the whole system of tangents to the surface at each point are principal axes, while at every point of the other not one of the whole system of tangents possesses that property.

This is easily conceivable in the case of the complete envelope itself, whatever be its nature; for since every principal axis which touches that surface must touch it at two points, in order to find the direction at any point on either of its sheets of the particular tangent, or the directions of all the particular tangents if there be more than one, which will possess the property of being principal axes, we have only to make that point the point of contact of a tangent plane to its own sheet of the envelope, and the vertex of a cone enveloping the other sheet, the cone obviously will always intersect the tangent plane in all the lines passing through the point which will have double contact with the surface, and therefore, *a fortiori*, in all the principal axes sought; and it is easy to see that if the particular determinate number of the intersecting sides which are principal axes be in general even, these particular sides may as often be all imaginary as real.

Now that number is always even, and moreover it can never exceed two; for, from every point of the envelope, or more generally from every point of any surface whatever given or arbitrarily assumed in the body, there diverges a cone of principal axes, which cone also will of course always intersect the tangent plane at each point of the surface in all the tangents at that point which are principal axes; but every cone of principal axes in a body wherever be its vertex (let this be assumed for the present) is always of the second order, and can therefore intersect any plane passing through its vertex in never more than two right lines, and these may as often be both imaginary as real.

Hence at no point on either sheet of the surface envelope of a system of principal axes restricted by a single condition can ever more than two axes of the system touch the surface, and hence also both sheets may possess regions for which no axis of the system will touch at any point whatever, in which case the curve or curves separating the available from the untouched regions will on each sheet possess of course the property that at all its points the two tangent principal axes will coincide with each other; the same, moreover, may exactly be said of any surface whatever given or arbitrarily assumed in the body, it possesses generally (though not universally) two distinct regions, for all the points on one of which two principal axes containing between them an angle of finite magnitude will touch the surface, and for all the points on the other of which no tangent whatever will be a principal axis,

the curve or curves separating these two regions of real and imaginary contact being the locus or loci of that system or systems of points on the surface for which the two tangent principal axes coincide with each other; the first of these regions on every surface obviously contains all those points for which the diverging cone of the second order of principal axes intersects the tangent plane in two real and different right lines; the second for the same reason contains all those for which the two intersecting sides are both imaginary, and the separating curve or curves is the locus of that particular system or systems of points for which the tangent plane to the surface is also a tangent plane to the cone of principal axes.

These two different regions always exist on every closed surface of any form which is of small dimensions in comparison with its distance from the center of gravity, for, at the different points of such a surface the diverging cone of principal axes varies but little in magnitude, position, and figure, while between the same limits the tangent plane passes through every possible variety of position.

Moreover, since for every point on such a surface there generally exists a second on the opposite side for which the tangent plane is parallel to that at the first point, the bounding curve separating the regions of real and imaginary contact, consists generally of two distinct closed curves, returning each into themselves, and dividing the surface into three distinct portions; hence on such a surface; the regions of the two species of contact consist generally, one of the two opposite and unconnected caps separated from each other by the intermediate interval, and the other of the continuous zone between them.

On the contrary, the region of real contact for the most part monopolises the whole of every closed surface which contains within it the centre of gravity and which is such as every direction to present its concavity towards that point; for at the different points of every such surface, the diverging cone of principal axis experiences considerable variations both in position and figure; while the corresponding tangent plane also goes simultaneously through every possible variety of position. Also on every such surface the separating curves are of course altogether imaginary, and moreover the two conjugate systems of curves so intimately connected with the distribution of the enveloping system of principal axes, viz. the lines of contact and the lines of regression of the two component

systems of developoble surfaces experience in their case considerable modifications from the general type, and become peculiarly simple in their nature and distribution.

But, in general, different surfaces assumed arbitrarily in a body present every imaginable variety both with respect to the nature and magnitude of the two different regions, and with respect to the nature and form of the two bounding curves; in some cases the region of real contact will take up the whole surface, in others the region of imaginary contact may occupy the whole of it; the two regions, either or both finite or extending to infinity in any or in every direction according to the nature of the assumed surface in each particular case, will in some cases be both continuous, while in others they will be discontinuous and consist each of two or more separate and detached portions of the surface, or they will be one continuous and the other consisting of several unconnected portions; and also, the curves of separation bounding the different regions will in some cases be altogether imaginary, while in others they will be one or both real, in some cases the real curve or curves will either or both consist of a single continuous curve finite or extending to infinity as the case may be, and in others, either or both will consist of two or more detached curves separated from each other by intervals of the surface and all closed or all open, or some closed and finite and others extending to infinity in either or in both directions; all of which will be perfectly manifest from a property to be presently established, viz. that two different curves of this nature distinct from each other in their respective properties exist on every surface whatever assumed arbitrarily in a body, and that they, and therefore with them the two regions of real and imaginary contact, are always determined by the intersections with that surface of two other determinable surfaces distinct also from each other and differing themselves in their nature and properties.

In the case of every surface, whatever be its nature, which is the complete envelope of a system of principal axes restricted by a single condition, there exist on each sheet two distinct curves of this nature, both possessing the property that at all their points the two tangent principal axes coincide with each other, and therefore both separating an untouched from an available region of that surface, but otherwise differing from each other in their nature and properties and arising each from a different cause. Of these two curves, thus independent of each other on the same

sheet, but each intimately connected with the corresponding curve on the other sheet, the two corresponding pairs possessing always each the same properties and being always together both real or both imaginary, one on each is the same for both sheets and is a portion of their common intersecting curve, the other on each is a portion of its curve of contact with their common circumscribing developable surface.

This there is no difficulty in seeing, for, at every point of the curve of contact with either sheet of the circumscribing developable common to the two sheets, the angle vanishes between two of the particular tangents which also touch the other sheet, that is, the two tangent principal axes all along the portion of that curve corresponding to those particular lines of double contact coincide with each other, and, at every point of the intersecting curve common to the two sheets, the angle between two of the tangents to either sheet, which also touch the other, is equal in magnitude to two right angles; hence at all points of the proper portion of that curve also the two tangent principal axes to either sheet coincide with each other.

Moreover, in the former case, the two coincident principal axes all along either curve of contact coincide obviously at each point of that curve with the corresponding edge of the circumscribing developable, that is with the tangent to the surface at that point conjugate to the tangent to the curve itself at the same point; and in the latter case, all along the common intersecting curve, they obviously coincide with the tangent itself at every point of that curve; hence on each sheet of the surface envelope of a system of principal axes restricted by a single condition, the two curves which separate the available from the untouched regions are not merely the loci of the two systems of points on that sheet for which the two tangent principal axes coincide with each other, but are moreover, one the envelope of the corresponding system of coincident axes themselves, and the other the envelope, not of its corresponding system of axes themselves but of the system of tangents to the surface conjugate to that system of axes.

The two developable surfaces generated by these two particular systems of axes, both surfaces of principal axes, and of principal axes which all touching the two sheets of the envelope belong to the system of axes enveloped by that surface, are in other respects also two very remarkable surfaces with respect to that system; for, notwithstanding that every system of principal axes subject to a single con-

dition may always be divided into two and not more than two different and distinct systems of developable surfaces, of which the surfaces of one will be all circumscribed to one sheet of the envelope and will have their arêtes de rebroussement all situated on the other, while the surfaces of the other system will be all circumscribed to the latter sheet and will have their arêtes de rebroussement all placed on the former. These two particular developable surfaces, the one circumscribed equally to the two sheets of the envelope, and the other having its arête de rebroussement equally on both, belong to neither of the above two systems, which are the only developable systems into which the original system of axes can be divided, and yet they are both developable surfaces of principal axes, and of principal axes which belong to that system.

This apparent paradox may be explained in almost exactly the same way as a particular solution of an ordinary differential equation has been generally explained, a solution which satisfies the equation, but which nevertheless does not belong to the only system or systems of solutions into which by the variation of the arbitrary constant or constants the complete integral of that equation may be always resolved, and in both cases the geometrical interpretations are almost precisely similar; for, as we shall just now see, these two developable surfaces of principal axes, though not belonging to either of the two component developable systems, contain each one or more of the edges of every individual surface of both those systems; and moreover, the two curves of contact with the two sheets of the enveloping surface of their common circumscribing developable are the envelopes each of the whole system of lines of contact of the component developable system circumscribed to its own sheet of that surface, while, at the same time, the common intersecting curve arête de rebroussement of the other developable surface is also the envelope common to the two sheets of the two systems of lines of regression, the arêtes de rebroussement of the two developable systems; the lines of regression of one component system and the lines of contact of the other lying always on the same sheet of the envelope and forming these two systems of curves conjugate to each other.

These properties there is no difficulty in establishing, for, assuming arbitrarily any tangent to the curve of intersection on the portion of that curve corresponding to the principal axes, that tangent may be made the basis of two

developable surfaces, one circumscribed to one of the sheets of the envelope and having its arête de rebroussement on the other, and the other circumscribed to the latter sheet and having its arête de rebroussement on the former, and each belonging therefore to one of the two different component developable systems: and in proceeding in both directions to complete these two developables, it is obvious that the two lines of regression will both touch the intersecting curve at the point of contact of the assumed tangent, while the two lines of contact will consist each of two branches both terminating abruptly at the same point, and (since the edges of every developable circumscribed to any surface and the corresponding tangents to its curve of contact are always pairs of conjugate tangents to the surface) both having the same tangent at that point, viz. the tangent on their own sheet conjugate to the assumed tangent to the curve of intersection, but lying themselves on opposite sides of that tangent, and both turning their convexities towards it; hence the two lines of contact of the two developable surfaces have each a cusp of the ramphoid species at the point of contact of the assumed tangent, the tangents at the two cusps being the two conjugates to that tangent on the two sheets of the envelope; and proceeding thus in the same way from tangent to tangent of the curve of intersection we see that the whole system of tangents to that curve may be made the bases of two different and distinct systems of developable surfaces, one having their system of arêtes de rebroussement all on one sheet of the envelope and being all circumscribed to the second, and the other having their system of arêtes de rebroussement all on the latter sheet and being all circumscribed to the former; and again, that the two systems of lines of regression will all touch at one or more points the curve of intersection, and that the two systems of lines of contact will have each one or more cusps of the ramphoid species all ranged on that same curve, the cuspal points of each pair of the latter lines being equal in number and coinciding with the points of contact of the corresponding pair of the former, and every pair of cuspal tangents being conjugate each on its own sheet of the envelope to the corresponding tangent common to the corresponding pair of lines of regression.

Again, assuming arbitrarily any side of the common circumscribing developable surface, on the portion of that surface corresponding to the principal axes, that side may similarly be made the common basis of two developable

surfaces, one circumscribed to one of the sheets of the envelope and having its arête de rebroussement on the second, and the other circumscribed to the latter sheet and having its arête de rebroussement on the former, and therefore, as in the other case, belonging each to one of the two component developable systems; and proceeding as before in both directions to complete these two developable surfaces, it is equally obvious that the two lines of contact will in this case touch each the curve of contact with its own sheet of the common circumscribing developable, while the two lines of regression will have each a cusp of the ramphoid species at the point where the corresponding line of contact touches its enveloping curve, the tangent of each cusp being, as before, conjugate to the corresponding tangent of the curve of contact. Hence in this case, proceeding from edge to edge, the whole system of edges of the common circumscribing developable may be made the bases of two different and distinct systems of developable surfaces, one having their system of arêtes de rebroussement all on one sheet of the envelope and being all circumscribed to the second, and the other having their system of arêtes de rebroussement all on the latter sheet and being all circumscribed to the former; and of these, the two systems of lines of contact will each envelope the curve of contact with its own sheet of the common circumscribing developable, every individual line of the system touching that curve in one or more points as the case may be, while the two systems of lines of regression will in this case have each one or more cusps of the ramphoid species ranged all on the same two curves, and on every individual line of either system being equal in number and coinciding with all the points of contact of the corresponding line of contact on the same sheet, the cuspal tangent, as before, being in all cases conjugate to the tangent of the corresponding curve of the other system with respect to the sheet of the envelope on which they both lie.

[*To be continued.*]

ON LOGARITHMIC INTEGRALS OF THE SECOND ORDER.
PART II.

By FRANCIS W. NEWMAN.

(*Continued from* p. 100).

$$\left.\begin{aligned}\textit{On the Integrals } \Lambda(x, \alpha) &= \tfrac{1}{2}\int_0 \frac{\log x\,(x - \cos\alpha)\,dx}{x^2 - 2x\cos\alpha + 1}\\ \lambda(x, \alpha) &= \tfrac{1}{2}\int_0 \log(x^2 - 2x\cos\alpha + 1)\,\frac{dx}{x}\end{aligned}\right\}$$

which are related by the equation $\Lambda(x, \alpha) + \lambda(x, \alpha) = \frac{1}{2}\log x \log X$, *if* X *stands for* $(x^2 - 2x\cos\alpha + 1)$.

§ I.—*Simplest cases of* Λ.

1. It was shewn in Part I. that all the integrals included in $\int F_1 x \log F_2 x\, dx$ are reducible to common forms in conjunction with three peculiar integrals, Lx, $\daleth x$, and $\Lambda(x, a)$, of which the last alone remains to be treated. We suppose a to be between 0 and π, unless the contrary is stated. We also generally suppose x to be positive. When it comes out negative in any formula, we can restore it by help of the identical equation

$$\Lambda(-x, a) = -\Lambda(x, \pi - a),$$

which subsists by virtue of the convention (already proposed) that $\log x$ is always to mean $\frac{1}{2}\log(x^2)$.

$$\text{Assuming } x = \frac{\sin\omega}{\sin(\omega + a)}, \text{ or } \tan\omega = \frac{x\sin a}{1 - x\cos a},$$

$$\text{we get } \sqrt{X} = \frac{\sin a}{\sin(\omega + a)};$$

$$\text{whence } \Lambda(x, a) = \int \log\frac{\sin(\omega + a)}{\sin\omega}\,.\,d\log\sin(\omega + a);$$

which we shall hereafter denote by $\chi(\omega, a)$; so that $\Lambda(x, a)$ and $\chi(\omega, a)$ are identical forms. This substitution is chiefly of use in enabling us to understand the nature of other transformations at which we shall arrive. For the present, when ω is named, it is supposed to bear this relation to x.

2. To find the *complete function* $\Lambda(1, a)$, which $= -\lambda(1, a)$.

$$\text{Since } \lambda(x, a) = \tfrac{1}{2}\int_0 \log X\,\frac{dx}{x},\quad \frac{d\lambda}{da} = \int_0 \frac{\sin a}{X}\,dx = \omega.$$

$$\text{Make } x = 1, \therefore \tan\omega = \frac{\sin a}{1 - \cos a} = \cot\tfrac{1}{2}a, \text{ or } \omega = \tfrac{1}{2}(\pi - a).$$

Integrate $\dfrac{d\lambda}{da} = \frac{1}{2}(\pi - a)$; $\therefore \lambda(1, a) = c + \frac{1}{2}\pi a - \frac{1}{4}a^2$.

To find c, make $a = 0$; $\lambda(x, 0) = \int_0 \log(1 - x)\dfrac{dx}{x} = L(1 - x)$,

whence $\lambda(1, 0) = L0$, or $c = -\frac{1}{6}\pi^2$,

$$\left.\begin{aligned} \therefore\ \Lambda(1, a) &= -\lambda(1, a) = \tfrac{1}{6}\pi^2 - \tfrac{1}{2}\pi a + \tfrac{1}{4}a^2. \\ \text{Hence also } \Lambda(1, \pi - a) &= \tfrac{1}{4}a^2 - \tfrac{1}{12}\pi^2. \end{aligned}\right\} \ldots\ldots(1).$$

3. To find Λ at special values of a.

At the extreme values, ($a = 0$ and $a = \pi$), X is an algebraic square, $(x \pm 1)^2$. Hence

$$\left.\begin{aligned} \Lambda(x, 0) &= Lx - L0 \\ \Lambda(x, \pi) &= L(-x) - L0 = lxl(1 + x) - L(1 + x) \end{aligned}\right\} \ldots(2).$$

In the following process, we for a moment suppose a to increase from 0 to any magnitude; and, to shew both variables, write $f(x, a)$ instead of X. Then, by a well-known formula of Trigonometry,

$$f(x^n, na) = f(x, a) . f\left(x, a + \frac{2\pi}{n}\right) . f\left(x, a + \frac{4\pi}{n}\right) \ldots f\left(x, a + \frac{2n-2}{n}\pi\right).$$

Differentiate logarithmically: multiply by $(2n)^{-1}.\log(x^n) = 2^{-1}\log x$, and integrate: then

$$\frac{1}{n}\Lambda(x^n, na) = \Lambda(x, a) + \Lambda\left(x, \ a + \frac{2\pi}{n}\right)$$
$$+ \Lambda\left(x, a + \frac{4\pi}{n}\right) + \ldots\ldots + \Lambda\left(x, a + \frac{2n-2}{n}\pi\right) \ldots\ldots(3).$$

In particular, if $n = 2$, $\cos(a + \pi) = \cos(\pi - a)$,

$$\left.\begin{aligned} \therefore\ \tfrac{1}{2}\Lambda(x^2, 2a) &= \Lambda(x, a) + \Lambda(x, \pi - a) \\ &= \Lambda(x, a) + \Lambda(-x, a) \end{aligned}\right\} \ldots\ldots(4),$$

which has a certain analogy to $Lx + L(-x) = \frac{1}{2}L(x^2) + \frac{3}{2}L0$.

When $a = \dfrac{\pi}{n}$, we have

$$\left.\begin{aligned} &\frac{1}{n}\Lambda(x^n, \pi) \text{ or } L(-x^n) - L0 \\ &= \Lambda\left(x, \frac{\pi}{n}\right) + \Lambda\left(x, \frac{3\pi}{n}\right) + \Lambda\left(x, \frac{5\pi}{n}\right) + \ldots + \Lambda\left(x, \frac{2n-1}{n}\pi\right) \end{aligned}\right\}.$$

Introvert the terms; and to make every a fall between 0 and π, observe that $\cos\dfrac{2n - r}{n}\pi = \cos\dfrac{r\pi}{n}$. Then we find that

$$\left.\begin{array}{l}\text{if } S = \Lambda\left(x, \dfrac{\pi}{n}\right) + \Lambda\left(x, \dfrac{3\pi}{n}\right) + \Lambda\left(x, \dfrac{5\pi}{n}\right) + \ldots\ldots \\ \text{when } n \text{ is even, } S \text{ to } \dfrac{n}{2} \text{ terms} = \dfrac{1}{2n}\Lambda(x^n, \pi), \\ \text{and when } n \text{ is odd,} \\ \qquad S \text{ to } \dfrac{n-1}{2} \text{ terms} = \dfrac{1}{2n}\Lambda(x^n, \pi) - \tfrac{1}{2}\Lambda(x, \pi)\end{array}\right\}\ldots(5).$$

In particular,

$$\left.\begin{array}{l}\text{If } n = 2, \quad \Lambda(x, \tfrac{1}{2}\pi) = \tfrac{1}{4}\Lambda(x^2, \pi): \\ \text{If } n = 3, \quad \Lambda(x, \tfrac{1}{3}\pi) = \tfrac{1}{6}\Lambda(x^3, \pi) - \tfrac{1}{2}\Lambda(x, \pi)\end{array}\right\}\ldots(6).$$

If in equation (3) we make $na = 2\pi$, and n is odd, we similarly have

$$\Lambda\left(x, \frac{2\pi}{n}\right) + \Lambda\left(x, \frac{4\pi}{n}\right) + \ldots + \Lambda\left(x, \frac{n-1}{n}\pi\right)$$

$$= \frac{1}{2n}\Lambda(x^n, 0) - \tfrac{1}{2}\Lambda(x, 0)\ldots\ldots(7):$$

If $n = 3$ in the last,

$$\Lambda(x, \tfrac{2}{3}\pi) = \tfrac{1}{6}\Lambda(x^3, 0) - \tfrac{1}{2}\Lambda(x, 0)\ldots\ldots\ldots(8).$$

Thus we know $\Lambda(x, a)$ in finite functions of x, by means of L, when a has any of the values $0, \pi, \tfrac{1}{2}\pi, \tfrac{1}{3}\pi, \tfrac{2}{3}\pi$. It will afterwards appear, that the assertion may be extended to the case of $a =$ any of these, *divided by* 2^n, when n is an arbitrary integer.

4. To find Λ, when x (at the upper limit) is a given function of a.

Generally, if $u = Fx$, $V = \psi(x, a)$, and $U = \int u\,dV$,

$$\frac{dU}{da} = \int u\frac{d^2V}{dx\,da}dx = \int u\frac{dV'}{dx}dx, \text{ if } V' = \frac{dV}{da};$$

Integrate by parts, and the last is $(uV' - \int V'du)$; and the *total* differential

$$d(U) = \frac{dU}{dx}dx + \frac{dU}{da}da = u\frac{dV}{dx}dx + \left(u\frac{dV}{da} - \int V'du\right)da$$

$$= u.d(V) - (\int V'du)\,da.$$

In the present case, let $u = \tfrac{1}{2}\log x$, $V = \log X$, and observe that $V' = 2xX^{-1}\sin a$, which vanishes with x; and $\int V'du = \omega$, which also vanishes with x, as $\dfrac{dU}{da}$ or (here) $\dfrac{d\Lambda}{da}$ ought to do

No constant then is needed in integrating; and we get

$$d(\Lambda) = \tfrac{1}{2}\log x.d\,(\log X) - \omega da \,\ldots\ldots\ldots (9)$$

for the total differential, when x is a function of a.

The extreme case of $x = (\cos a)^{\circ} = 1$, reproduces equation (1), as it ought.

Assume $x = 2\cos a$, $X = 1$, $\tan\omega = -\tan 2a$, $\omega = \pi - 2a$; $\therefore d\Lambda = -(\pi - 2a)\,da$. Observe that Λ vanishes when $x = 0$, and consequently (here) when $a = \frac{1}{2}\pi$, and we get

$$\Lambda \text{ or } \Lambda(2\cos a, a) = (\tfrac{1}{2}\pi - a)^2 \,\ldots\ldots (10).$$

This is the most remarkable equation we have yet met.

It may also be denoted (if $a = \cos a$) by

$$\int_0^{2a} \frac{\log x(x-a)}{x^2 - 2ax + 1}\,.\,dx = (\sin^{-1} a)^2.$$

Once more, assume $x = \cos a = a$, $\therefore X = 1 - a^2$, $\tan\omega = \cot a$, $\omega = \frac{1}{2}\pi - a$; whence

$$d\Lambda = \tfrac{1}{2}\log a\,d.\log(1 - a^2) - (\tfrac{1}{2}\pi - a)\,da,$$

$$\therefore \Lambda = \tfrac{1}{4}L(a^2) + \tfrac{1}{2}(\tfrac{1}{2}\pi - a)^2 + \text{const};$$

$$\text{or } \Lambda(\cos a, a) = \tfrac{1}{4}L(\cos^2 a) + \tfrac{1}{2}(\tfrac{1}{3}\pi^2 - \pi a + a^2)\ldots.(11):$$

$$= \tfrac{1}{4}L(\cos^2 a) + \Lambda(1, a) + \tfrac{1}{4}a^2.$$

The constant is determined as before, by observing that Λ must vanish when $a = \frac{1}{2}\pi$; also $L0 = -\frac{1}{6}\pi^2$.

There is an infinity of other assumptions ($x = Fa$) which make (9) integrable in finite terms. Again, ω may be expanded in a series of cosines or sines of multiples of a, after which we may try to integrate. But the cases in which these processes succeed appear to be more easily treated by some of the methods which follow.

§ II.—*On changing* $\Lambda(x, a)$ *to* $\Lambda(y, a)$.

5. As long as x is small (say $x < \pm\frac{1}{2}$), we may develop $\log X$ by the well-known series, and obtain

$$-\lambda(x, a) = \frac{x\cos a}{1^2} + \frac{x^2\cos 2a}{2^2} + \frac{x^3\cos 3a}{3^2} + \&\text{c.}\ldots.(12),$$

from which $\lambda(x, a)$ and $\Lambda(x, a)$ are found. But when x is not small enough, we may try to reduce $\Lambda(x, a)$ to $\Lambda(y, a)$, in which y is small.

For the present, let Y stand for $y^2 - 2y\cos a + 1$.

6. Then, *first*, put $xy = 1$, $\therefore$ $Y = Xx^{-2}$, $\log y = -\log x$, $d \log Y = d \log X - 2d \log x$;

$$\Lambda x + \Lambda y = \tfrac{1}{2}\int lx(dlX - dlY) = \int lx\, d\, lx$$
$$= \tfrac{1}{2}\log^2 x + \text{const};$$
$$\text{or}\quad \Lambda x + \Lambda x^{-1} = 2\Lambda 1 + \tfrac{1}{2}\log^2 x \ldots\ldots\ldots (13),$$

which serves to reduce Λx, when x is > 1, to Λy, where y is < 1.

Cor. 1. When $x = \infty$, Λx converges to $2\Lambda 1 + \frac{1}{2}\log^2 x$.

Cor. 2. As $Lx + Lx^{-1} = \frac{1}{2}\log^2 x$,

$$\therefore (\Lambda x - Lx) + (\Lambda x^{-1} - Lx^{-1}) = 2\Lambda 1.$$

Secondly, suppose $x + y = 2\cos a$, $X = 1 - xy = Y$;

$$\therefore\ \Lambda x + \Lambda y = \tfrac{1}{2}\int (lx + ly)\, dl\,(1 - xy)$$
$$= \tfrac{1}{2}\int l(xy).dl(1 - xy) = \tfrac{1}{2}L(xy) + C:$$

$$\left.\begin{aligned}\text{whence } \Lambda x + \Lambda y - \Lambda(2\cos a) &= \tfrac{1}{2}L(xy) - \tfrac{1}{2}L0,\\ \text{when } x + y &= 2\cos a.\end{aligned}\right\}\ \ldots (14).$$

As we know $\Lambda(2\cos a)$ from equation (10), this enables us to find Λx by means of Λy, whenever x is near to $2\cos a$. It also gives

$$2\Lambda\cos a - \Lambda(2\cos a) = \tfrac{1}{2}L\cos^2 a - \tfrac{1}{2}L0,$$

which is verified by (10) and (11).
Make $x = 1$, and we find

$$\Lambda(2\cos a - 1) = \tfrac{1}{2}L(2\cos a - 1) + \tfrac{1}{6}\pi^2 - \tfrac{1}{2}3\pi a + \tfrac{1}{4}5a^2 \ldots (14^*).$$

We may combine the two last integrations, by supposing $x^{-1} + y = 2\cos a$, which gives

$$\Lambda x - \Lambda y + \tfrac{1}{2}a^2 = \tfrac{1}{2}\log^2 x - \tfrac{1}{2}L\frac{y}{x}\ \ldots\ldots (15).$$

Again, in this write y^{-1} for y, and eliminate Λy^{-1} by means of (13), and $L(x^{-1}y^{-1})$ by means of the known properties of L; then

$$\left\{\begin{aligned}&x^{-1} + y^{-1} = 2\cos a;\\ &\Lambda x + \Lambda y - \pi\left(\tfrac{1}{3}\pi - a\right) = \tfrac{1}{2}L(xy) + \tfrac{1}{4}\log^2\left(\frac{x}{y}\right)\end{aligned}\right\}\ \ldots (16).$$

But in this, the arbitrary constant is liable to change by reason of discontinuity, if x or y passes through zero.

7. The four suppositions here made have something in common. In (13), (14), and (16), we find

$$\frac{dx}{X} = \frac{dy}{Y};\ \text{and in (15)},\ \frac{dx}{X} = -\frac{dy}{Y}.$$

Let us in all suppose $x = \dfrac{\sin \omega}{\sin (\omega + a)}$;

then if $y = \dfrac{\sin \theta}{\sin (\theta - a)}$, $xy = 1$, when $\theta = \omega + a$

......, but $x + y = 2 \cos a$, when $\theta = \omega + 2a$.

8. By equation (13) we can obtain $\Lambda (\sec a)$ and $\Lambda (\frac{1}{2} \sec a)$ from $\Lambda (\cos a)$ and $\Lambda (2 \cos a)$. Observing that

$$L \cos^2 a + L \sec^2 a = 2 \log^2 \cos a,$$

we have
$$\left.\begin{aligned} \Lambda \sec a &= \tfrac{1}{4} L \sec^2 a + \tfrac{1}{2}\pi (\tfrac{1}{3}\pi - a) \\ \Lambda (\tfrac{1}{2} \sec a) &= \tfrac{1}{2} \log^2 (\tfrac{1}{2} \sec a) + \tfrac{1}{12}\pi^2 - \tfrac{1}{2} a^2 \end{aligned}\right\} \ldots .(17).$$

9. Farther, since we fulfil the relation $x^{-1} + y = 2 \cos a$, by supposing

$$x = \frac{\sin \omega}{\sin (\omega + a)}, \quad y = \frac{\sin (\omega - a)}{\sin \omega},$$

it is evident that if Λx is known, we can by the repeated use of (15) find $\Lambda \dfrac{\sin \{\omega - (n+1) a\}}{\sin (\omega - na)}$. Or conversely, if Λy is known, we can deduce $\Lambda \dfrac{\sin (\omega + na)}{\sin \{\omega + (n+1) a\}}$.

For example: *first*, let m_n stand for $\dfrac{\cos na}{\cos (n-1) a}$; then $m_n^{-1} + m_{n+1} = 2 \cos a$;

$$\therefore\ 2\Lambda m_{n+1} - 2\Lambda m_n = a^2 + L (m_n^{-1} m_{n+1}) - \log^2 m_n.$$

For n write 1, 2, 3....$(n-1)$, and add the results, taking Λm_1 from (11);

$$\left.\begin{aligned} \therefore\ 2\Lambda \frac{\cos na}{\cos (n-1) a} &= \tfrac{1}{3}\pi^2 - \pi a + na^2 \\ &+ \tfrac{1}{2} L \cos^2 a + L (m_1^{-1} m_2) + L (m_2^{-1} m_3) + \ldots + L (m^{-1}_{n-1} m_n) \\ &- \log^2 m_1 - \log_2 m_2 - \&\text{c.} \ldots\ldots - \log^2 m_{n-1} : \end{aligned}\right\}$$

where for $\cos^2 a$ we may write $(m_0^{-1} m_1)$.

Similarly, if $m_n = \dfrac{\sin na}{\sin (n+1) a}$, $m_n^{-1} + m_{n-1} = 2 \cos a$; and Λm_1 is known by (17); so that

$$2\Lambda \frac{\sin na}{\sin (n+1) a} = \tfrac{1}{6}\pi^2 - na^2 - L \frac{m_1}{m_2} - \ldots . - L \frac{m_{n-1}}{m_n}$$
$$+ \log^2 m_1 + \log^2 m_2 + \ldots . + \log^2 m_n.$$

10. A more general relation between Λx and Λy is obtainable by Mr. Fox Talbot's Method of Symmetrical Integrals.

Let $X = (m - x)\, v$, $Y = (m - y)\, v$; where m is constant, and x, y functions of v. Then x and y are the two roots of

$$x^2 - (2 \cos \alpha - v)\, x + (1 - mv) = 0\,;$$

consequently $x + y = 2 \cos \alpha - v$, $xy = 1 - mv$;

and, eliminating v,

$$(m - x)(m - y) = 1 - 2m \cos \alpha + m^2 = M.$$

Now $\Lambda x + \Lambda y = \frac{1}{2}\int lx\, \{dlv + dl(m - x)\} + \frac{1}{2}\int ly\, \{dlv + dl(m - y)\}$

$$= \tfrac{1}{2}\int l(xy)\, dlv + \tfrac{1}{2}\int lx\, dl(m - x) + \tfrac{1}{2}\int ly\, dl(m - y).$$

The first integral

$$= \int l(1 - mv)\, dlv = L(1 - mv) = L(xy)\,;$$

the second

$$= lml(m - x) + L\frac{x}{m}\,; \text{ the third} = lml(m - y) + L\frac{y}{m}.$$

Observe that $lml(m - x) + lml(m - y) = lmlM = \text{const.}$;

$$\therefore\ 2\Lambda x + 2\Lambda y - \text{const.} = L(xy) + L\frac{x}{m} + L\frac{y}{m}.$$

Let $x = e$, when $y = 0$; then $e = 2 \cos \alpha - m^{-1}$; so that Λe and Λm will be known from one another by (15). Also

$$\left(1 - \frac{x}{m}\right)\left(1 - \frac{y}{m}\right) = 1 - \frac{e}{m} = 1 - 2e \cos \alpha + e^2 = E.$$

Finally

$$2\,(\Lambda x + \Lambda y - \Lambda e) = L(xy) + L\frac{x}{m} + L\frac{y}{m} - L\frac{e}{m} - 2L0 \ldots (18).$$

By slightly varying the integration, we get

$$2\,(\lambda x + \lambda y - \lambda e)$$

$$= L(1 - xy) + L\left(1 - \frac{x}{m}\right) + L\left(1 - \frac{y}{m}\right) - L\left(1 - \frac{e}{m}\right) \ldots (18^*),$$

which may be convenient when m, or one of the variables, is negative.

By giving special values to m, such as make Λm and Λe known functions, the equations become available to us in many ways. But in order to understand our result, it will be well to transform it by means of ω and the function χ of Art. 1.

11. First observe that as m is arbitrary, it may be so settled (x being given), as to assign any required value to y. This amounts to making x and y independent variables, and m a function of them. And in fact,

$$m = \frac{1 - xy}{2 \cos a - (x + y)},$$

by which m may be eliminated, and e expressed in terms of x and y.

But, leaving m constant, put

$$x = \frac{\sin \omega}{\sin (\omega + a)}, \quad y = \frac{\sin \theta}{\sin (\theta + a)}, \quad e = \frac{\sin \eta}{\sin (\eta + a)};$$

$$\text{Now } d\omega + d\theta = \frac{\sin a\, dx}{X} + \frac{\sin a\, dy}{Y} = \frac{\sin a}{v}\left(\frac{dx}{m - x} + \frac{dy}{m - y}\right)$$

$$= \frac{-\sin a}{v}\, d \log \{(m - x).(m - y)\} = \frac{-\sin a}{v}\, d \log M = 0;$$

$\therefore \omega + \theta = \text{const.}$

When $\omega = 0$, $x = 0$; $\therefore y = e$, and $\theta = \eta$; or const. $= \eta$ and

$$\omega + \theta = \eta.$$

Again, $m^{-1} = 2 \cos a - e = \dfrac{\sin (\eta + 2a)}{\sin (\eta + a)}$. To save room, let $\psi(\omega)$ denote the function of ω to which x is equal;

$$\left.\begin{aligned} \therefore 2\{\chi\omega + \chi\theta - \chi(\omega + \theta)\} &= L(\psi\omega.\psi\theta) - 2L0 \\ + L\frac{\psi\omega}{\psi(\omega + \theta + a)} + L\frac{\psi\theta}{\psi(\omega + \theta + a)} &- L\frac{\psi(\omega + \theta)}{\psi(\omega + \theta + a)} \end{aligned}\right\} \ldots (19):$$

in which m and e or η have been eliminated, and ω, θ remain as independent variables.

Thus, if $\chi\omega$ and $\chi\theta$ are known for any particular values of ω and θ, $\chi(\omega + \theta)$ may be hence deduced. It is at once evident, that $\chi(\omega - \theta)$, $\chi(n\omega)$ and in fact $\chi\left(\dfrac{m}{n}\,\omega\right)$ may also be found, in finite terms, and by equations of the first degree. Yet this is rather a theoretical truth, than one of utility for calculation.

Since we already know Λx for the values $x = 1$, $x = 2 \cos a$, $x^{-1} = 2 \cos a$, $x = \cos a$, $x^{-1} = \cos a$; which correspond to $\omega = \frac{1}{2}(\pi - a)$, $\omega = \pi - 2a$, $\omega = a$, $\omega = \frac{1}{2}\pi - a$, $\omega = \pm \frac{1}{2}\pi$; we may start from any of these; and by addition or subtraction obtain a variety of results, and many more by combining division. If, however, we seek for $\chi(na)$ in this way, the result is far more complicated than in Art. 9. The cases

which may chiefly deserve to be pointed out as attainable, are the following:

$\omega = \pi - a$	$\omega = \frac{1}{2}\pi + a$	$\omega = \frac{1}{2}(\pi - 3a)$.
$\omega = \frac{1}{2}a$	$\omega = \frac{1}{4}\pi \pm \frac{1}{2}a$	$\omega = \frac{1}{4}\pi - a$.
$\omega = \pm \frac{1}{4}\pi$	$\omega = \frac{1}{4}(\pi \pm a)$	

The process of bisection, transferred to (18), consists in supposing Λe known, and making $x = y = m(1 \mp \sqrt{E})$, then $2\Lambda x$ is found from Λe.

As equation (18) is virtually an integral of the equation

$$\frac{dx}{X} + \frac{dy}{Y} = 0,$$

and contains an additional arbitrary constant m; no more general result is attainable in this direction. If we had assumed $X = v(m - x)(n - y)$, $Y = v(m - y)(n - y)$, the integral of $d(\Lambda x + \Lambda y)$, hence arising, would be a mere combination of (18) with (15), and would tell us nothing new.

§ III.—*On changing* $\Lambda(x, a)$ *to* $\Lambda(y, \beta)$.

12. The properties hitherto attained have more show of utility than they make good, in regard to the general reduction of $\Lambda(x, a)$ to another calculable function. In that respect more advantage is derived from changing a in Λ simultaneously with x.

With a view to this, put now Y for $y^2 - 2y\cos\beta + 1$.

For a first integration, *assume* $x\cos a + y\cos\beta = 1$, and $a + \beta = \frac{1}{2}\pi$;

$$\therefore\ X = \sin^2 a\,(x^2 + y^2),\ Y = \sin^2\beta\,(x^2 + y^2);$$

$$dlX = dlY = dl\left(1 + \frac{x^2}{y^2}\right) + dl.y^2.$$

$$\text{Also}\quad \frac{dx}{\sin a} + \frac{dy}{\sin\beta} = 0;\quad \frac{\sin a\,dx}{X} + \frac{\sin\beta\,dy}{Y} = 0.$$

$$\text{Now } \Lambda(x, a) - \Lambda(y, \beta) = \tfrac{1}{2}\textstyle\int(lx - ly)\,dl(x^2 + y^2)$$

$$= \frac{1}{2}\int l\frac{x}{y}.dl\left(\frac{x^2}{y^2} + 1\right) + \int l\frac{x}{y}\,dly.$$

The former integral $= \frac{1}{4}L(-x^2y^{-2})$. In the latter, observe that $(xy^{-1}) = \tan a\,\{(y\cos\beta)^{-1} - 1\}$. Put $y\cos\beta = v^{-1}$, $dly = -dlv$,

$$\therefore \textstyle\int l(xy^{-1})\,dly = -\int\{l\tan a + l(v - 1)\}\,dlv$$

$$= l\tan a.l(y\cos\beta) - L(1 - v) + \text{const.}$$

To correct, make $x = 0$, $y = \sec\beta$; then, since $v - 1 = \dfrac{x\cos\alpha}{y\cos\beta}$,

$$\left.\begin{array}{l}\Lambda(x, \alpha) - \Lambda(y, \beta) + \Lambda(\sec\beta, \beta) \\ = \frac{1}{4}L\left(-\dfrac{x^2}{y^2}\right) - L\left(-\dfrac{x\cos\alpha}{y\cos\beta}\right) + l\tan\alpha ly\cos\beta + \frac{3}{4}L0\end{array}\right\} \dots (20).$$

To avoid the negatives under L, we may use the equation

$$L(-z) = lzl(1+z) - L(1+z) + L0;$$

reducing by which, we get for the right-hand member

$$\tfrac{1}{2}l(xy^{-1}).lX - \tfrac{1}{4}L(Xy^{-2}\cos^{-2}\beta) + L(y\cos\beta)^{-1}\dots\text{(20*)}.$$

This may be named the *Complementary Equation.*† If in it we make $\beta = \frac{1}{3}\pi$, $\Lambda(y, \beta)$ is a known function of y. Hence we can by it determine $\Lambda(x, \frac{1}{6}\pi)$ as a known function of x.

If in the last, $x = \dfrac{\sin\omega}{\sin(\omega+\alpha)}$, $y = \dfrac{\sin\theta}{\sin(\theta+\beta)}$, then from $\dfrac{\sin\alpha dx}{X} + \dfrac{\sin\alpha dy}{Y} = 0$, we get $d\omega + d\theta = 0$, or $\omega + \theta = \text{const.}$ Let $y = 0$, $x\cos\alpha = 1$; $\sin\omega\cos\alpha = \sin(\omega+\alpha)$, or $\omega = \frac{1}{2}\pi$, and $\theta = 0$;

$$\therefore \omega + \theta = \tfrac{1}{2}\pi; \quad \alpha + \beta = \tfrac{1}{2}\pi; \quad \sin(\omega+\alpha) = \sin(\theta+\beta).$$

13. For a second integration, *assume*

$$x = 2y\cos\beta - 1, \quad \alpha = 2\beta;$$

$$\therefore X = 4Y\cos^2\beta, \quad dlX = dlY, \quad dx = 2dy\cos\beta;$$

$$\frac{\sin\alpha\, dx}{X} = \frac{2\sin\alpha\, dy\cos\beta}{4Y\cos^2\beta} = \frac{\sin\beta dy}{Y},$$

which gives $d\omega = d\theta$. When $\omega = 0$, $x = 0$, $y^{-1} = 2\cos\beta$, $\tan\theta = 2y\sin\beta$, or $\theta = \beta$; $\therefore \omega = \theta - \beta$; or $\omega + \alpha = \theta + \beta$. So much for the algebraic relations of the variables.

$$\text{Now } \Lambda(x, \alpha) - 2\Lambda(y, \beta) = \tfrac{1}{2}\int\log(xy^{-2})\,dlY.$$

Let $y^{-1} = z$, $Z = z^2 - 2z\cos\beta + 1$; whence $Y = Zy^2$,

$$xy^{-2} = 1 - Z = (2y\cos\beta - 1)y^{-2};$$

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† In many formulas, it would appear that if our tables of Lx gave $\log x$ rather than x for the *argument*, this would be more convenient in the application. This suggests that *the same* table might give x, hyp $\log x$ and Lx.

$$\therefore \tfrac{1}{2}\int l(xy^{-2})\, dlY = \tfrac{1}{2}\int l(1 - Z)\, dlZ + \int \log \frac{2y\cos\beta - 1}{y^2}\, dly$$

$$= \tfrac{1}{2}L(1 - Z) + L(1 - 2y\cos\beta) - \log^2 y.$$

Hence

$$\Lambda(x, \alpha) - 2\Lambda(y, \beta) = \tfrac{1}{2}L(xy^{-2}) + L(-x) - \log^2 y + C \ldots (21)^*.$$

To find C, let $x = 0$, $y = \frac{1}{2}\sec\beta$; and after a few reductions, $C = \frac{1}{4}\alpha^2 + \frac{1}{12}\pi^2$. The same result is found by making $x = -1$, $y = 0$; but the infinites under L and *log* are a little more troublesome.

This may be called the *Equation of Bisection*, since $\alpha = 2\beta$, and since $\Lambda(x, \alpha) - 2\Lambda(y, \beta)$ is expressed in known functions.

It follows that if $\Lambda(x, \alpha)$ is a known function of x for some one value of α, so is $\Lambda(x, \frac{1}{2}\alpha)$. For we have only to make $x' = 2x\cos\frac{1}{2}\alpha - 1$, and determine $\Lambda(x, \frac{1}{2}\alpha)$ from $\Lambda(x', \alpha)$ by equation (21).

Hence $\Lambda\left(x, \dfrac{\pi}{2^n}\right)$ and $\Lambda\left(x, \dfrac{\pi}{3.2^n}\right)$ can be obtained in finite terms as known functions of x, since we know $\Lambda(x, \frac{1}{2}\pi)$ and $\Lambda(x, \frac{1}{3}\pi)$.

14. By a repeated use of the equation of bisection, it is evident that $\Lambda(x, \alpha)$ is reducible to $\Lambda(x_n, 2^{-n}\alpha)$, which, when $n = \infty$, is $\Lambda(x_n, 0)$ a known function. It may be worth while to enter into a few details concerning this.

Let α_n represent $2^{-n}\alpha$, and from x suppose $x_1, x_2 \ldots$ to be derived by the law

$$2x_1\cos\alpha_1 = 1 + x;\ 2x_2\cos\alpha_2 = 1 + x_1;\ \&c.\ \&c.\ldots.$$

It is easy to compute these by the intervention of ω. For we had

$$\omega + \alpha = \theta + \beta \text{ or } = \omega_1 + \alpha_1 = \omega_2 + \alpha_2 = \omega_3 + \alpha_3 = \&c.,$$

$$\text{whence } \omega_n = \omega + \alpha - \alpha_n.$$

Thus $x_n = \dfrac{\sin(\omega + \alpha - 2^{-n}\alpha)}{\sin(\omega + \alpha)}$, which, when $n = \infty$, converges to 1, and nearly $= 1 - 2^{-n}\alpha\cot(\omega + \alpha)$. (We must entirely except the case of $\omega + \alpha = 0$ or $= \pi$, which gives $x = \infty$.) Hence $2^n.\Lambda(x_n\ \alpha_n) = 2^n.\Lambda(x_n\ 0) = 2^n.\{Lx_n + \frac{1}{6}\pi^2\} = 2^n.(x_n - 1) + 2^n.\frac{1}{6}\pi^2 = -\alpha\cot(\omega + \alpha) + 2^n.\frac{1}{6}\pi^2$. Apply equation (21) n times: multiply the results by $2^0, 2^1, 2^2 \ldots 2^{n-1}$, and add all together. Substitute for $2^n.\Lambda(x_n\ \alpha_n)$ as above, and be careful to note that $2^{-2} + 2^{-3} + \ldots + 2^{-\infty} = \frac{1}{2}$,

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* It is easy to combine equations (20), (21) with (13).

and $(2^0 + 2^1 + 2^2 + \dots + 2^{n-1})\frac{1}{12}\pi^2 + 2^n.\frac{1}{6}\pi^2$

$$= \tfrac{1}{6}\pi^2 + (2^0 + 2^1 + \dots + 2^{n-1})\tfrac{1}{4}\pi^2 ;$$

after which, making $n = \infty$, we get

$$\left.\begin{aligned}\Lambda(x, a) = &\tfrac{1}{2}a^2 - a\cot(\omega + a) + \tfrac{1}{6}\pi^2 \\ &+ 2^0\{\tfrac{1}{4}\pi^2 + L(-x) - \log^2 x_1 + \tfrac{1}{2}L(x\,x_1^{-2})\} \\ &+ 2^1\{\tfrac{1}{4}\pi^2 + L(-x_1) - \log^2 x_2 + \tfrac{1}{2}L(x_1 x_2^{-2})\} \\ &+ 2^2\{\tfrac{1}{4}\pi^2 + L(-x_2) - \log^2 x_3 + \tfrac{1}{2}L(x_2 x_3^{-2})\} \\ &+ \&c.\ \&c.\dots\dots\end{aligned}\right\}\dots(22).$$

This always converges, yet not rapidly. When x_n is approaching its limit 1, we may approximately determine the remnant of the series, by the formulas

$$\left.\begin{aligned}&L(1-h) = -h - \tfrac{1}{4}h^2 - \tfrac{1}{9}h^3;\ \tfrac{1}{4}\pi^2 + L(-1+h) = \tfrac{1}{4}h^2 + \tfrac{1}{6}h^3;\ \text{when } h \text{ is very small.}\\ &\log^2(1-h) = h^2 + h^3.\ \text{Also } h_n = a_n\{(1 - \tfrac{1}{6}a_n^2)\cot(\omega + a) + \tfrac{1}{2}a_n\};\\ &\text{and } h_{n+1} = (\tfrac{1}{2}h_n - \tfrac{1}{8}a_n^2)(1 + \tfrac{1}{8}a_n^2),\ \text{very nearly.}\end{aligned}\right\}$$

But the great defect of the method is, that even if we start with x nearly $= 1$, we still do not any the more rapidly reach the limit $x_n = 1$: hence the series has no practical interest, unless indeed at once both x is very near to 1, and cos a between x and 1, a case which is the most troublesome of all in the method of Art. 16.

15. The equation of bisection would farther enable us to increase the number of functions $x = Fa$, which give $\Lambda(x, a)$ as a known function of a. For let $x = Fa$ be any one function, for which $\Lambda(x, a)$ is known; put $2x_1 \cos\frac{1}{2}a = 1 + x$, or put $x' = 2x\cos a - 1$; and $x_1\ x'$ are new functions of a, for which $\Lambda(x_1, \frac{1}{2}a)$ and $\Lambda(x', 2a)$ are known.

If we could integrate so as to obtain $\Lambda(x, a) + m\Lambda(y, \beta)$ in known functions, when $d\omega \propto d\theta$, by means of some *general* relations uniting a, β, m; it would more than anything else perfect what is wanting in this theory.

§ IV.—*To calculate* $\Lambda(x, a)$ *in any case.*

16. We have now the means of reducing $\Lambda(x, a)$ in all cases to another function $\Lambda(x', a')$ in which x' shall be less than $\frac{1}{2}$; which will enable us to apply equation (12).

Avoiding details, it will suffice here to shew the possibility of the transformation.

First, when $\alpha > 60°$; if x is > 1, we may reduce Λ to the case of $x < 1$ by equation (13). If then the new x is between $\frac{1}{2}$ and 1, put $x' = -x$, $\alpha' = \pi - \alpha = 2\beta$, $x' = 2y \cos\beta - 1$; $\Lambda(x, \alpha) = \Lambda(x', \alpha')$. Apply the equation of bisection to reduce $\Lambda(x, \alpha)$ to $\Lambda(y, \beta)$. Now as α' is $< 120°$, β is $< 60°$, $2\cos\beta > 1$, $2y\cos\beta > y$; $\therefore 1 + x'$ or $1 - x > y$, or $y < \frac{1}{2}$.

Next, when α is $< 30°$, put $\alpha + \beta = \frac{1}{2}\pi$, and use the Complementary Equation. Since β is $> 60°$, this case is reduced to the former.

Thirdly, when α is between 60° and 30°. Here x is by hypothesis between 1 and $\frac{1}{2}$, and $2\cos\alpha$ between $\sqrt{3}$ and 1; so that $2x\cos\alpha$ is between $\sqrt{3}$ and $\frac{1}{2}$.—We separate the case of $x > \cos\alpha$; in which we can proceed exactly as when α was $> 60°$. For since $2y\sin\frac{1}{2}\alpha = 1 - x$, which is $< 1 - \cos\alpha$ or than $2\sin^2\frac{1}{2}\alpha$, $\therefore y$ is $< \sin\frac{1}{2}\alpha < \frac{1}{2}$.—When x is *not* $> \cos\alpha$, $2x\cos\alpha$ does not exceed $2\cos^2\alpha$ or $\frac{3}{2}$; so that its limits are $\frac{3}{2}$ and $\frac{1}{2}$. Put $y = 2x\cos\alpha - 1$, and y is between $+\frac{1}{2}$ and $-\frac{1}{2}$. If then $2\alpha = \beta$, we can reduce by equation (21), only exchanging x with y, and α with β.

The simplicity of the coefficients in equation (12), which are known by common tables, would lead us to prefer that series when other things are equal. Yet if x is near to $\frac{1}{2}$, its convergence is not such as to give accuracy to many decimal places without great labour; and some of the following methods may become preferable.

§ V.—*To take advantage of α lying within certain limits.*

17. If α is extremely small, and x is $< \frac{1}{2}$; or if, x being near to 1, the product $2\sin\frac{1}{2}\alpha \,.\left(\dfrac{x}{1-x}\right)$ is still very small.

Put $b = 2\sin\frac{1}{2}\alpha$, $z = \dfrac{x}{1-x}$, or $x = \dfrac{z}{1+z}$; $1 - x = \dfrac{1}{1+z}$;

$$X = (1-x)^2 + b^2x = (1-x)^2.\{1 + b^2z.(1+z)\}$$

$$d\log x = d\{\log z - \log(1+z)\} = \frac{dz}{z(1+z)};$$

$$\therefore \lambda(x, \alpha) = \tfrac{1}{2}\textstyle\int_0 \log X\, d\log x = \int_0 \log(1-x)\, d\log x$$
$$+ \tfrac{1}{2}\textstyle\int_0 \log\{1 + b^2z.(1+z)\}\,\dfrac{dz}{z(1+z)}$$
$$= L(1-x) + \tfrac{1}{2}P \ldots\ldots\ldots\ldots\ldots\ldots\ldots (23),$$

if $P = \int_0\{b^2 - \frac{1}{2}b^4\, z.(1+z) + \frac{1}{3}b^6\, z^2.(1+z)^2 - \&c.\ldots\}\, dz$

$$= b^2z - \tfrac{1}{2}b^4(\tfrac{1}{2}z^2 + \tfrac{1}{3}z^3) + \tfrac{1}{3}b^6(\tfrac{1}{3}z^3 + 2\,\tfrac{1}{4}z^4 + \tfrac{1}{5}z^5) - \&c.\ldots (23^*),$$

which converges rapidly, since bz is very small.

18. If, on the contrary, a is very near to π (which is always the more favourable case, x being supposed positive), let $x = \tan^2 \frac{1}{2}\omega$; then $\lambda(x, a) = L(1 + x) - 2\Omega$,

$$\text{if } \Omega = -\tfrac{1}{2}\int_0 \log\left(1 - \cos^2 \tfrac{1}{2}a \sin^2 \omega\right) \frac{d\omega}{\sin \omega}.$$

If we develop the logarithm, we readily see that Ω may take the form

$$A_0 - 2A_1 \cos \omega + 2A_3 \frac{\cos 3\omega}{3} - 2A_5 \frac{\cos 5\omega}{5} + \&c.$$

To find A_0, let $\omega = \frac{1}{2}\pi$, $\Omega = A_0$, $x = 1$, $\therefore \lambda(1, a) = L2 - 2A_0$; whence $2A_0 = \dfrac{\pi^2}{12} + \Lambda(1, a) = \left(\dfrac{\pi - a}{2}\right)^2$.—Let $\pi - a = 4\beta$,

$$\therefore A_0 = 2\beta^2.$$

Next $\dfrac{d\Omega}{d\omega} = 2A_1 \sin \omega - 2A_3 \sin 3\omega + 2A_5 \sin 5\omega - \&c.$,

$$\text{also } \frac{d\Omega}{d\omega} = -\tfrac{1}{2}\log\left(1 - \sin^2 2\beta \,.\, \sin^2 \omega\right) \frac{1}{\sin \omega}.$$

Put $b = \tan \beta$, $\sin 2\beta = \dfrac{2b}{1 + b^2}$; and the value of $\sin \omega \,.\, \dfrac{d\Omega}{d\omega}$

$$\text{is } \log(1 + b^2) - \tfrac{1}{2}\log(1 + 2b^2 \cos 2\omega + b^4),$$

or $\log(1 + b^2) - b^2 \cos 2\omega + \frac{1}{2}b^4 \cos 4\omega - \frac{1}{3}b^6 \cos 6\omega + \&c.$, which is to be made equal to

$$2 \sin \omega \left\{A_1 \sin \omega - A_3 \sin 3\omega + A_5 \sin \omega - \&c.\right\},$$

or $A_1(1 - \cos 2\omega) - A_3(\cos 2\omega - \cos 4\omega) + A_5(\cos 4\omega - \cos 6\omega) - \&c.$

Hence we get $\quad A_1 = \log(1 + b^2)$,

$$A_1 + A_3 = \frac{b^2}{1}; \text{ and generally } A_{2n-1} + A_{2n+1} = \frac{b^{2n}}{n}.$$

In the First Part of these investigations we have used $\phi_n x$ to denote $\int_0 \tan^{n-1} x \, dx$; which yields $\phi_1 x = x$, $\phi_2 x = \frac{1}{2}\log(1 + \tan^2 x)$ or $\log \sec x$; and $\phi_n x + \phi_{n+2} x = \dfrac{\tan^n x}{n}$.

Thus $A_1 = 2\phi_2\beta$; $A_3 = 2\phi_4\beta$; $A_5 = 2\phi_6\beta$; &c. . . .

$$\left.\begin{aligned} \text{and } \lambda(x, a) = L(1 + x) - 4\beta^2 + 8\phi_2\beta \,.\, &\frac{\cos \omega}{1} - 8\phi_4\beta \,.\, \frac{\cos 3\omega}{3} \\ + 8\phi_6\beta \,.\, &\frac{\cos 5\omega}{5} - 8\phi_8\beta \,.\, \frac{\cos 7\omega}{7} \\ + \&c. \ldots &\&c. \ldots \end{aligned}\right\} \ldots (24),$$

which converges best when β is least, or a nearest to π.

19. To find Λ and λ, when α is near to $\frac{1}{2}\pi$.

$$\text{Put } x = \tan\left(\tfrac{1}{4}\pi - \tfrac{1}{2}\omega\right);$$

$$\therefore \lambda(x, \alpha) = \tfrac{1}{4}L(1 + x^2) - \Omega, \text{ if } \Omega = \tfrac{1}{2}\int \log(1 - \cos\alpha\cos\omega)\frac{d\omega}{\cos\omega}.$$

Assume $\Omega = C - C_0\omega - 2C_1\sin\omega - 2C_2\dfrac{\sin 2\omega}{2} - 2C_3\dfrac{\sin 3\omega}{3} - \&c.$

To determine C, put $\omega = 0$, $x = 1$, $\Omega = C$, $\lambda(1, \alpha) = \frac{1}{4}L2 - C$,

$$\text{or } C = \tfrac{1}{4}L2 + \Lambda(1, \alpha) = \tfrac{1}{16}3\pi^2 - \tfrac{1}{2}\pi\alpha + \tfrac{1}{4}\alpha^2.$$

To determine C_0,

$$\text{we have } -\frac{d\Omega}{d\omega} = C_0 + 2C_1\cos\omega + 2C_2\cos 2\omega + \&c.\ldots$$

Multiply by $d\omega$, and integrate from $\omega = 0$ to $\omega = \pi$, observing that $\int_0^\pi \cos n\omega\, d\omega = 0$, for all integer values of n; also $\int_0^\pi C_0 d\omega = \pi C_0$.

Then $\pi C_0 = \displaystyle\int_0^\pi -\frac{d\Omega}{d\omega}d\omega = \int_1^{-1} -\frac{d\Omega}{dx}dx = \Omega$ (from $x = -1$ to $x = 1$) $= \{\frac{1}{4}L2 - \lambda(1, \alpha)\} - \{\frac{1}{4}L2 - \lambda(-1, \alpha)\} = \Lambda(1, \alpha) - \Lambda(1, \pi - \alpha) = (\frac{1}{6}\pi^2 - \frac{1}{2}\pi\alpha + \frac{1}{4}\alpha^2) - (\frac{1}{4}\alpha^2 - \frac{1}{12}\pi^2) = \frac{1}{4}\pi^2 - \frac{1}{2}\pi\alpha.$

Whence $C_0 = \frac{1}{2}(\frac{1}{2}\pi - \alpha)$. Call this γ. $\therefore C = \frac{1}{2}\pi\gamma + \gamma^2$.

$$\text{Farther, put } c = \tan\gamma,\ \cos\alpha = \sin 2\gamma = \frac{2c}{1 + c^2};$$

$$\cos\omega.\frac{d\Omega}{d\omega} = \tfrac{1}{2}\log\left(1 - \frac{2c\cos\omega}{1 + c^2}\right)$$

$$= -\tfrac{1}{2}\log(1 + c^2) - c\cos\omega - \tfrac{1}{2}c^2\cos 2\omega - \tfrac{1}{3}c^3\cos 3\omega - \&c.\ldots$$

$$\text{But } -\cos\omega\frac{d\Omega}{d\omega} = \cos\omega\left\{C_0 + 2C_1\cos\omega + 2C_2\cos 2\omega + \&c.\ldots\right\}$$

$$= C_1 + (C_0 + C_2)\cos\omega + (C_1 + C_3)\cos 2\omega + (C_2 + C_4)\cos 3\omega + \&c.\ldots$$

$$\therefore C_1 = \tfrac{1}{2}\log(1 + c^2) = \phi_2\gamma;\quad C_2 = c - C_0 = \tan\gamma - \gamma = \phi_3\gamma;$$

$$C_3 = \tfrac{1}{2}c^2 - C_1 = \phi_4\gamma;\qquad C_4 = \tfrac{1}{3}c^3 - C_2 = \phi_5\gamma;\ \&c.\ldots$$

$$\left.\begin{aligned}\text{Whence } \lambda(x, \alpha) &= \tfrac{1}{4}L(1 + x^2) - (\tfrac{1}{2}\pi\gamma + \gamma^2) + \gamma\omega \\ &+ 2\phi_2\gamma.\frac{\sin\omega}{1} + 2\phi_3\gamma.\frac{\sin 2\omega}{2} + 2\phi_4\gamma.\frac{\sin 3\omega}{3} + \&c.\end{aligned}\right\}\ldots(25),$$

where $\gamma = \frac{1}{4}\pi - \frac{1}{2}\alpha$, $x = \tan(\frac{1}{4}\pi - \frac{1}{2}\omega)$.

The convergence is rapid when α is very near to $\frac{1}{2}\pi$.

In equation (24), put $\omega = 0$, $\Omega = 0$;

$$\therefore \tfrac{1}{2}\beta^2 = \phi_2\beta - \tfrac{1}{3}\phi_4\beta + \tfrac{1}{5}\phi_6\beta - \&c.$$

In the value of $\dfrac{d\Omega}{d\omega}$ corresponding, make $\omega = \frac{1}{2}\pi$;

$$\therefore -\tfrac{1}{4}\log\cos 2\beta = \phi_2\beta + \phi_4\beta + \phi_6\beta + \&c.\ldots$$

In equation (25), if we change γ into $-\gamma$, $\phi_{2n}\gamma$ remains unchanged, and $\phi_{2n-1}\gamma$ changes sign. By adding the two results thus obtained, we might easily reproduce equation (24).

Put $\omega = \pi$ in the value of $\frac{d\Omega}{d\theta}$ corresponding to (25);

$$\therefore \tfrac{1}{2}\log(1 + \sin 2\gamma) = \phi_1\gamma - 2\phi_2\gamma + 2\phi_3\gamma - 2\phi_4\gamma + \&c.,$$

$$\text{so } \tfrac{1}{2}\log(1 - \sin 2\gamma) = -\phi_1\gamma - 2\phi_2\gamma - 2\phi_3\gamma - 2\phi_4\gamma - \&c.;$$

which gives not only

$$-\tfrac{1}{4}\log\cos 2\gamma = \phi_2\gamma + \phi_4\gamma + \phi_6\gamma + \&c.,$$

but also $\frac{1}{2}\log\tan(\frac{1}{4}\pi + \gamma) = \phi_1\gamma + 2\phi_3\gamma + 2\phi_5\gamma + \&c.\ldots.$

These are mere properties of the functions $\phi_1, \phi_2, \phi_3 \ldots$ and can in several ways be verified.

The series (24), (25) cannot be practically used with advantage, unless we have tables of $\phi_n a$; but these might be computed with so much ease, within the limits $a = 0$, $a = 45°$, that this is apparently the best method of adding completeness to this branch of the calculus. The following section will shew that the use of ϕ_n is not confined to the particular cases contemplated in equations (24), (25).

§ VI.—*To find* Λ, *when* x *is near to* 1.

20. We shall suppose a to be $< 90°$, and deal with

$$\Lambda(x, \pi - a) \text{ and } \Lambda(x, a) \text{ separately.}$$

Put $\cos a = \frac{1 - m^2}{1 + m^2}$, or $m = \tan\frac{1}{2}a$; $X' = 1 + 2x\cos a + x^2$;

$$\therefore (1 + m^2)X' = (1 + x)^2 + m^2(1 - x)^2. \quad \text{Let } y = \frac{1 - x}{1 + x}.$$

Then $\Lambda(x, \pi - a) = \frac{1}{2}\log x \log X' - L(1 + x) + R$,

$$\text{if } R = \int \log\frac{1 + m^2y^2}{1 + m^2} \cdot \frac{dy}{1 - y^2}.$$

$$\text{Assume } -\frac{dR}{dy} = M_0 - M_2y^2 + M_4y^4 - \&c.\ldots;$$

$$\therefore \log(1 + m^2) - \frac{m^2y^2}{1} + \frac{m^4y^4}{2} - \frac{m^6y^6}{3} + \&c.\ldots.$$

$$= (1 - y^2)\{M_0 - M_2y^2 + M_4y^4 - \&c.\ldots.\},$$

which gives

$M_0 = \log(1 + m^2) = 2\phi_2\frac{1}{2}a$; $M_2 = 2\phi_4\frac{1}{2}a$; $M_4 = 2\phi_6\frac{1}{2}a$; &c....

To find the constant after integrating $\frac{dR}{dy}$, make $x = 1$,

$$y = 0\,;\ \therefore\ R = \text{const.} = \Lambda(1, \pi - a) + L2 = \tfrac{1}{4}a^2.$$

Hence
$$\left.\begin{array}{l}\Lambda(x, \pi - a) = \tfrac{1}{2}\log x \log X' - L(1 + x) + \tfrac{1}{4}a^2 \\ \quad - 2\phi_2\,\tfrac{1}{2}a \,.\, \dfrac{y}{1} + 2\phi_4\,\tfrac{1}{2}a\,.\,\dfrac{y^3}{3} - 2\phi_6\,\tfrac{1}{2}a\,.\,\dfrac{y^5}{5} + \&c.\ldots\end{array}\right\}\ \ldots(26).$$

When $x = 0$, $y = 1$,

$$\tfrac{1}{2}\,.\,\tfrac{1}{4}a^2 = \phi_2\,\tfrac{1}{2}a - \tfrac{1}{3}\,\phi_4\,\tfrac{1}{2}a + \tfrac{1}{5}\,\phi_6\,\tfrac{1}{2}a - \&c.\ldots\ldots$$

which serves to verify the conclusion.

21. Next, observe that $(1 + m^2)\,.\,X = (1 + x)^2.(m^2 + y^2)$, so that

$$\Lambda(x, a) = \left(\frac{\pi - a}{2}\right)^2 + \log x \log(1 + x) - L(1 + x) - S \ldots(27),$$

$$\text{if}\ \ S = \tfrac{1}{2}\int_0 \log\frac{1 + y}{1 - y}\,.\,d\log(m^2 + y^2)\,;$$

where the arbitrary constant is found, as before, by making $x = 1$.

Developing $\log\dfrac{1 + y}{1 - y}$,

$$\tfrac{1}{2}S = \textstyle\int_0(y + \tfrac{1}{3}y^3 + \tfrac{1}{5}y^5 + \&c.\ldots)\,\dfrac{y\,dy}{m^2 + y^2}$$

$$= \int_0\frac{y^2\,dy}{m^2 + y^2} + \frac{1}{3}\int_0\frac{y^4\,dy}{m^2 + y^2} + \frac{1}{5}\int_0\frac{y^6\,dy}{m^2 + y^2} + \&c.\ldots\ldots$$

Add $\tfrac{1}{2}a\tan^{-1}\dfrac{y}{m} = \displaystyle\int_0\frac{m^2\,dy}{m^2 + y^2} - \frac{1}{3}\int_0\frac{m^4\,dy}{m^2 + y^2} + \frac{1}{5}\int_0\frac{m^6\,dy}{m^2 + y^2} - \&c.\ldots$

and the sum is $\int_0 dy - \tfrac{1}{3}\int_0(m^2 - y^2)\,dy + \tfrac{1}{5}\int_0(m^4 - m^2y^2 + y^4)\,dy - \&c.$

$$= \textstyle\int_0(M_1 + M_3y^2 + M_5y^4 + \ldots)\,dy,$$

$$\begin{aligned}\text{if}\ \ M_1 &= 1 - \tfrac{1}{3}m^2 + \tfrac{1}{5}m^4 - \tfrac{1}{7}m^6 + \&c.\ldots\ldots = m^{-1}.\,\tfrac{1}{2}a,\\ M_3 &= \tfrac{1}{3} - \tfrac{1}{5}m^2 + \tfrac{1}{7}m^4 - \tfrac{1}{9}m^6 + \&c.\ \ldots\ldots = m^{-3}.\,\phi_3\,\tfrac{1}{2}a,\\ M_5 &= \tfrac{1}{5} - \tfrac{1}{7}m^2 + \tfrac{1}{9}m^4 - \tfrac{1}{11}m^6 + \&c.\ldots\ldots = m^{-5}.\,\phi_5\,\tfrac{1}{2}a,\end{aligned}$$

and so on.

Write $z = ym^{-1} = \dfrac{1}{m}\,.\,\dfrac{1 - x}{1 + x}$; then we finally get

$$S = -a\tan^{-1}z + 2\left\{\phi_1\,\tfrac{1}{2}a\,.\,\frac{z}{1} + \phi_3\,\tfrac{1}{2}a\,.\,\tfrac{1}{3}z^3 + \phi_5\,\tfrac{1}{2}a\,.\,\tfrac{1}{5}z^5 + \&c.\ldots\right\}\ldots(27^*).$$

The multipliers m^{-1}, m^{-3}, $m^{-5}\ldots$ here injure the convergence, as compared with that of (26). Yet in the worst case, $m = 0$,

the coefficients M_1, M_3, M_5. . . .become 1, $\frac{1}{3}$, $\frac{1}{5}$. . . .so that the series always converges faster than

$$y + 3^{-2}y^3 + 5^{-2}y^5 + \ldots .\&c.\ldots .$$

and when x is $> \frac{1}{2}$, y^2 is $< \frac{1}{9}$; which is a far better convergence than we ordinarily get from equation (12).

22. We may increase the convergence (for the latter case only) by representing the given function as $\Lambda(x^2, 2a)$, and using the formula

$$\tfrac{1}{2}\Lambda(x^2, 2a) = \Lambda(x, \pi - a) + \Lambda(x, a),$$

taking $\Lambda(x, \pi - a)$ from (26) and $\Lambda(x, a)$ from (27). Thus if we wish to estimate $\Lambda(h, \mu)$, where h and μ are given, put $x^2 = h$, $2a = \mu$; then $y = \dfrac{1 - \sqrt{h}}{1 + \sqrt{h}}$, which is smaller than if we had made $x = h$, or $y = \dfrac{1 - h}{1 + h}$. In fact, this will enable us to restrict the use of equation (12) to the case of $x < \frac{1}{4}$; for if the variable is $> \frac{1}{4}$, call it x^2; $\therefore y^2$ is $< \frac{1}{9}$, and we find Λ by combining (26) and (27).

Supposing tables of ϕ_n to have been formed, it would perhaps be worth while, for the sake of the method just suggested, to add to them the values of f_n; where

$$\begin{array}{l|l} f_1 a = \cot a\, \phi_1 a + \phi_2 a & 3f_3 a = \cot^3 a\, \phi_3 a - \phi_4 a \\ 5f_5 a = \cot^5 a . \phi_5 a + \phi_6 a & 7f_7 a = \cot^7 a\, \phi_7 a - \phi_8 a \\ \&c.\ldots & \&c.\ldots \end{array}$$

whence we obtain

$$\left.\begin{aligned} &\tfrac{1}{2}\Lambda(x^2, 4a) = a^2 + (\tfrac{1}{2}\pi - a)^2 \\ &+ lxl(1 + x) + \tfrac{1}{2}lxl(1 + 2x \cos 2a + x^2) - 2L(1 + x) \\ &+ 2a . \tan^{-1}(y \cot a) - 2\{yf_1 a + y^3 f_3 a + y^5 f_5 a + \ldots\} \end{aligned}\right\} \ldots (28).$$

This is more compact to the eye: yet we here lose the advantage of regularity in the decrease of the coefficients.

§ VII.—*To find Λ when x is near to $\cos a$.*

23. When x is near to $2 \cos a$, we may reduce $\Lambda(x, a)$ by means of equation (14); but no such property has occurred with reference to $\cos a$.

Let $y = 1 - \dfrac{x}{\cos a}$; $X = (y \cos a)^2 + \sin^2 a$;

$$dlX = dl(y^2 + \tan^2 a).$$

Put $y \cos a = v$, $x^2 = (\cos a - v)^2 = (\cos^2 a - v^2) \div \dfrac{\cos a + v}{\cos a - v}$;

$$\therefore\ 2lx = l(\cos^2 a - v^2) - l\left(\frac{1+y}{1-y}\right).$$

$$\left.\begin{aligned} \text{Say } T &= \tfrac{1}{4}\int l(\cos^2 a - v^2)\, dl\,(v^2 + \sin^2 a);\\ U &= \tfrac{1}{4}\int_0 l\left(\frac{1+y}{1-y}\right) dl\,(y^2 + \tan^2 a); \end{aligned}\right\} \therefore \Lambda(x, a) = c + T - U.$$

Let $\cos^2 a - v^2 = V$, $v^2 + \sin^2 a = 1 - V$, $T = \frac{1}{4}\int l V\, dl\,(1 - V) = \frac{1}{4}L(V)$. To find c, let $y = 0$, $\therefore \Lambda(\cos a, a) = c + \frac{1}{4}L \cos^2 a$, or $c = \frac{1}{6}\pi^2 - \frac{1}{2}\pi a + \frac{1}{2}a^2$. Observe also that $V = 1 - X$.

To find U, we have only to compare it with $\frac{1}{2}S$ of Art. 21, and write $\tan a$ for m; that is, a for $\frac{1}{2}a$. Hence if $u = y \cot a = (\cos a - x) \operatorname{cosec} a$,

$$\left.\begin{aligned} \Lambda(x, a) &= (\tfrac{1}{6}\pi^2 - \tfrac{1}{2}\pi a + \tfrac{1}{2}a^2) + \tfrac{1}{4}L(2x \cos a - x^2)\\ &\quad + a \tan^{-1} u + \frac{u}{1}\phi_1 a + \frac{u^3}{3}\phi_3 a + \frac{u^5}{5}\phi_5 a + \&\text{c}\ldots \end{aligned}\right\} (29).$$

§ VIII.—*Geometrical idea of the function* $y = \Lambda(x, a)$.

24. When a is $< 90°$, $\dfrac{dy}{dx}$ or $\dfrac{\log x \,.\, (x - \cos a)}{X}$ is positive from $x = 0$ to $x = \cos a$, and then negative until $x = 1$; after which it is perpetually positive. Thus Λx increases up to $\Lambda \cos a$, which is a maximum, and decreases down to $\Lambda 1$, which is (geometrically) a minimum. But it is not certainly a numerical minimum, if it has become negative.

Since $\Lambda 1 = \left(\dfrac{\pi - a}{2}\right)^2 - \dfrac{\pi^2}{12}$, this cannot be negative, unless $(\pi - a)^2 < \frac{1}{3}\pi^2$, or $a > (1 - 3^{-\frac{1}{2}})\,\pi$, which brings a near to the limit $\frac{1}{2}\pi$. If a is $< (1 - 3^{-\frac{1}{2}})\,\pi$, Λx never becomes negative; and $\Lambda 1$ is a numerical minimum.

When $x = 0$, $\dfrac{dy}{dx} = -\log x \,.\, \cos a = +\infty$ when a is $< 90°$, or is $-\infty$ when a is $> 90°$. When $x = \cos a$, or $x = 1$, $\dfrac{dy}{dx} = 0$.

Again, the curve has an infinite branch corresponding to $x = \infty$, which gives $\Lambda x = 2\Lambda 1 + \frac{1}{2}\log^2 x$.

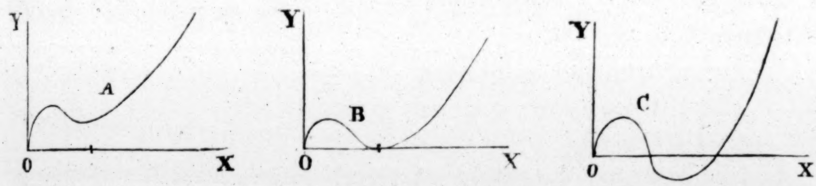
When a is $< 90°$, but so little less as to make $\Lambda 1$ negative, there are *two* values of x (one on each side of $x = 1$), such as to make $\Lambda x = 0$; besides the value $x = 0$.

When a is $> 90°$, $\dfrac{dy}{dx}$ is negative from $x = 0$ to $x = 1$, after which it is always positive; and, as before, Λx is positive infinity when $x = \infty$. There is then *one* value x that makes

$\Lambda x = 0$, besides $x = 0$. This value of x is > 1, since $\Lambda 1$ is now essentially negative.

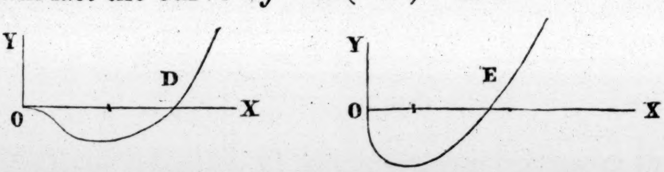
When $a = \frac{1}{2}\pi$ exactly, $\frac{dy}{dx} = \frac{x \log x}{1 + x^2}$; which vanishes with x. A curve is thus produced essentially different from, and intermediate to, the other two species. Up to $x = 1$, $\frac{dy}{dx}$ is negative; and afterwards positive.

This suffices to give a rough notion of the three species of curves with which we are here concerned; viz. *first*, when a is $< 90°$, we have forms such as *A, B, C.*



Of these, the form *B* exhibits $\Lambda 1$ exactly $= 0$; which gives $a = \pi(1 - 3^{-\frac{1}{2}})$. In *A*, a is greater; and in *C*, a is less than this limiting value.

Secondly, when $a = 90°$, the form appears to be such as *D*. This is in fact the curve $4y = L(-x^2) - L0$.



But *Thirdly*, when a is $> 90°$, we have the form *E*, which appears to be much simpler than the others.

ON A PROBLEM IN COMBINATIONS.*

By the Rev. Thomas P. Kirkman, A.B., Rector of Croft with Southworth, Lancashire.

If $Q_{x,y,z}$ denote the greatest number of combinations of y together, that can be made with x symbols, so that no combination of z together shall be twice employed; $Q_{x,y,1}$ is the greatest integer in $\frac{x}{y}$, and $Q_{x,y,y}$ is $\frac{x^{y|-1}}{y^{y|-1}}$, or the number of combinations of y together that can be made with x things. Thus division of integers, and this simple problem of com-

* Read before the Literary and Philosophical Society of Manchester, December 15, 1846.

binations, would appear to be particular cases of the more general problem, whose solution is $Q_{x,y,z}$.

The object of this paper is to assign $Q_{x,3,2}$; and to establish the following theorem:

If Q_x denote the greatest number of triads that can be formed with x symbols, so that no duad shall be twice employed, then

$$3Q_x = x\frac{x-1}{2} - V_x,$$

if for V_x we put $6k+4$, when $x = 6n-1$; $\frac{1}{2}x + 3k + 1$, when $x = 6n-2$; 0, when $x = 6n+1$ or $6n+3$; and $\frac{1}{2}x$, when $x = 6n$ or $6n+2$: where $2^m(2k+1) = n$; x, n, m, k, being all integers $\overline{\gtrless} 0$.

Q_x being defined as above,
let V_x be the number of duads possible with x things, that are excluded from Q_x.

Let Q_x' denote a system of triads formed with x things, in which no duad is twice employed, Q_x' being not of necessity a maximum, but such that v_x is the number of duads possible with x things, not employed in Q_x'.

The duads $bc\ cd\ d\ldots.r\ rs\ sb$, n duads in number, are a circle of n, or of n duads.

$(v_x = C_n)$ signifies, v_x can be made a circle of n; or Q_x' can be so formed, that v_x shall be left out, a circle of n duads.

$$D_x = x\frac{x-1}{2}, = \text{the duads possible with } x \text{ things.}$$

It will create no inconvenience if we speak of V_x, Q_x, and D_x either as numbers, or as visible arrangements of things.

(A) Prop. If $V_x = 0$, $V_{2x+1} = 0$ and $(v_{2x-1} = C_{2x-2})$ $[x = 2n+1]$.

Q_x being formed of triads made with the x letters $ABC\ldots$, let D_{x+1} be written out as follows, the $x+1$ letters being $a\ b\ c\ d\ldots\ldots$ Consider the line XX' as a circle continuous at XX', and divided into x parts:

X $\qquad\qquad\qquad\qquad$ X'

beginning at X, write out ab, ac, ad, &c., proceeding always to the right and in alphabetical order, and placing a duad under every division, as you proceed round the circle, except when it is necessary to omit a column (or division) in order to avoid repeating a letter in that column. By this means D_{x+1}, $(x+1 = 2m)$, will be written out in x columns, each containing all the $x+1$ letters $a, b, c, d\ldots.$ and no duad will be twice written, nor will any column contain the same letter twice. This is the simplest of the various rules that may be

given for writing out D_{x+1} in x columns, so as to answer the conditions imposed. Before these x columns, in any order, place the x letters $A, B, C. \ldots$ in succession as initial letters, completing with each letter (A) a column of $\frac{1}{2}(x+1)$ new triads. Add the x columns, of $\frac{1}{2}(x+1)$ triads each, to those already formed with $A, B, C. \ldots$ in Q_x, and the sum will be Q_{2x+1}; for, since all the duads possible with A, B, C, &c., and all those possible with a, b, c, &c., have been exhausted, and since each of the letters A, B, C, &c. is combined with each of a, b, c, &c., $V_{2x+1} = 0$.

Collect now all the triads in which a and b are found, and erase the letters a and b. The duads that remain after that erasure will be a circle of $2x-2$ duads, and the triads remaining, containing only $2x-1$ letters, are Q'_{2x-1}. It would be absurd to multiply symbols in order to demonstrate this result; as the operation is merely mechanical, and proves itself on the first trial. An example will presently be given.

(B) Prop. If $(v_{x+1} = C_x)$, $V_{2x+1} = 0$,

and $v_{2x-1} = C_{2x-2})$ $[x = 2m]$.

Let $BC, CD, DE, E. \ldots U, UB$ be the circle of x duads, and let A be the $(x+1)^{\text{th}}$ letter in Q'_{x+1}, which is supposed to be formed with the $x+1$ letters $A, B, C\ldots$ Let $a, b, c\ldots t$ be other x letters. Write out as before D_x $(x = 2m)$ in $x-1$ columns each containing all the x letters $a, b, c\ldots$ Erase now the x duads $ab, bc, cd, d. \ldots s, st, ta$, except the two last, st and ta, and write these $x-2$ duads in two additional columns, thus: placing the duads st, ta as erased duads below those two columns, (st) under ab, and (ta) under bc,*

$$\begin{array}{cc} ab & bc \\ cd & de \\ ef & fg \\ \vdots & \vdots \\ (st) & (ta) \end{array}$$

$D_x = \frac{1}{2}x + x\,\frac{1}{2}(x-2)$, is now written out in $x+1$ columns, one column containing $\frac{1}{2}x$ duads, in which all the x letters occur, and x columns containing each $(\frac{1}{2}x - 1)$ duads, and in each of which all the letters except two, viz. those in the erased duad, are found.

Using as a key, $\quad BCDE. \ldots TU,$

$\quad\quad\quad\quad\quad\quad\quad a\,b\,c\,d \ldots . s\,t,$

* Where (st), (ta) denote that st and ta are erased.

from the x triads following:

$$aBC, bCD, cDE, dEF \ldots\ldots sTU, tUB.$$

Place now A as initial letter to the column of $\frac{1}{2}x$ duads; and A is thus combined with all the x letters $a, b, c \ldots\ldots$ Place B as initial letter to that column of $\frac{1}{2}x - 1$ duads from which ta has been erased; C as initial to that column from which ab has been erased; D as initial to that from which bc has been erased; and so on. By this process, each of the letters $B, C, D \ldots U$ completes a column of $\frac{1}{2}x - 1$ triads, and is thus combined with all the x letters $a, b, c \ldots$ The new triads, in number $= x + D_x$, being added to Q'_{x+1}, will complete it into Q_{2x+1}; and $V_{2x+1} = 0$; because all the duads possible with the $2x + 1$ letters $A, B, C \ldots a, b, c \ldots$ have been exhausted.

The first of the new triads beginning with A will always be Aas, s being the $(x - 1)^{\text{th}}$ of the x letters $a, b, c \ldots\ldots$ Collect now all the triads in which a and s are found; erase a and s; and the duads remaining will always be a circle of $2x - 2$ duads. This process being, like the last, merely mechanical, needs no demonstration besides inspection of an example.

E. g. Q_3 is completed into Q_7 by the addition of D_4, and Q_7 into Q_{15} by the addition of D_8, as follows.

$$Q_7 = ABC$$

$$\begin{array}{lll} Aab & Bac & Cad \\ Acd & Bbd & Cbc. \end{array}$$

D_8 may be written according to the rule given page 192, thus:

hi	*hk*	*hl*	*hm*	*hn*	*ho*	*hp*
—	—	*ik*	*il*	*im*	*in*	*io*
—	*ip*	—	—	*kl*	*km*	*kn*
ko	—	—	*kp*	—	—	*lm*
ln	*lo*	—	—	—	*lp*	—
—	*mn*	*mo*	—	—	—	—
mp	—	—	*no*	—	—	—
—	—	*np*	—	*op*	—	—:

or it may be thus written:

hi	*hk*	*hl*	*hm*	*hn*	*ho*	*hp*
kl	*il*	*ik*	*in*	*im*	*ip*	*io*
mn	*mo*	*mp*	*ko*	*kp*	*mk*	*nk*
op	*np*	*no*	*lp*	*lo*	*nl*	*ml*,

which is done by a rule that could easily be assigned, and is the more symmetrical of the two. But symmetry has little to do with these combinations: they are essentially unsymmetrical.

Q_{15} =	ABC						
	ADE	BDF	CDG				
	AFG	BEG	CEF				
	Aab	Bac	Cad	Dae	Eaf	Fag	Gah
	Acg	Bbh	Cbc	Dbd	Ebe	Fbf	Gbg
	Adf	Bdg	Ceg	Dch	Ecd	Fce	Gcf
	Aeh	Bef	Cfh	Dfg	Egh	Fdh	Gde.

Collecting the triads in which a and b occur, we have

$$\begin{array}{ccccccc} Aab & Bac & Cad & Dae & Eaf & Fag & Gah \\ & Bbh & Cbc & Dbd & Ebc & Fbf & Gbg; \end{array}$$

and, erasing a and b, we obtain

$$Bc\ \ cC\ \ Cd\ \ dD\ \ De\ \ eE\ \ Ef\ \ fF\ \ Fg\ \ gG\ \ Gh\ \ hB,$$

a circle of 12 duads.

Q'_{13} is completed into Q_{25} as follows. First, Q'_{13} is written in capitals:

$A'B'C'$						
$A'D'E'$	$B'D'F'$	$C'D'G'$				
$A'F'G'$	$B'E'G'$	$C'E'F'$				
$A'CG$	$B'DG$	$C'EG$	$D'CH$	$E'CD$	$F'CE$	$G'CF$
$A'DF$	$B'EF$	$C'FH$	$D'FG$	$E'GH$	$F'DH$	$G'DE$
$A'EH$	$B'C$	$C'C$	$D'D$	$E'E$	$F'F$	$G'G$
	$B'H$	$C'D$	$D'E$	$E'F$	$F'G$	$G'H$.

Next, D_{12} is written thus:

C	F	C'	F'	D	G	D'	G'	E	A	E'	H	B'
(ab)	ac	ad	ae	af	ag	ah	ai	ak	al	am	ab	bc
cl	bm	(bc)	bd	be	bf	bg	bh	bi	bk	bl	cd	de
dk	dl	el	cm	(cd)	ce	cf	cg	ch	ci	ck	ef	fg
ei	ek	fk	fl	gl	dm	(de)	df	dg	dh	di	gh	hi
fh	fi	gi	gk	hk	hl	il	em	(ef)	eg	eh	ik	kl
gm	(gh)	hm	(hi)	im	(ik)	km	(kl)	lm	fm	(fg)	(lm)	(ma)

Using the key

$$\begin{array}{cccccccccccc} B' & C & C' & D & D' & E & E' & F & F' & G & G' & H \\ a & b & c & d & e & f & g & h & i & k & l & m, \end{array}$$

from the triads

$$aB'C \quad bC'C \quad cC'D \quad dDD' \quad eD'E \quad fEE' \quad gE'F$$
$$hFF' \quad iF'G \quad kGG' \quad lG'H \quad mHB';$$

then complete each of the above columns into triads by the addition of the letter marked over it, and Q_{25} is formed.

ABC													
ADE	BDF	CDG											
AFG	BEG	CEF											
ACG	BDG	CEG	DCH	ECD	FCE	GCF							
ADF	BEF	CFH	DFG	EGH	FDH	GDE							
AEH	BCa	CCb	DDd	EEf	FFh	GGk							
Aal	BHm	CDc	DEe	EFg	FGi	GHl							
Abk	Bbc	Cad	Dah	Eam	Fae	Gai	Hab	Eak	Daf	Cel	Fac	Gag
Aci	Bde	Cel	Dbg	Ebl	Fbd	Gbh	Hcd	Ebi	Dbe	Cdk	Fbm	Gbf
Adh	Bfg	Cfk	Dcf	Eck	Fcm	Gcg	Hef	Ech	Dgl	Cei	Fdl	Gce
Aeg	Bhi	Cgi	Dil	Edi	Ffl	Gdf	Hgh	Edg	Dhk	Cfh	Fek	Gdm
Afm	Bkl	Chm	Dkm	Eeh	Fgk	Gem	Hik	Elm	Dim	Cgm	Ffi	Ghl

And it is plain that $V_{25} = 0$.

If now the triads be collected in which the letters a and l occur, and those letters be erased, we obtain the following circle of 22 duads,

$$B'C \; Cc \; cF \; Fd \; dC' \; C'e \; eF' \; F'f \; fD \; Dg \; gG \; Gh$$
$$hD' \; D'i \; iG' \; G'H \; Hb \; bE' \; E'm \; mE \; Ek \; kB',$$

which proves that $(v_{23} = C_{22})$. In the same way this may be verified for any number.

Having established the two fundamental propositions,

$$\text{If } V_x = 0, \qquad V_{2x+1} = 0, \quad \text{and } (v_{2x-1} = C_{2x-2};$$
$$\text{If } (v_{x+1} = C_x), \quad V_{2x+1} = 0, \quad \text{and } (v_{2x-1} = C_{2x-2};$$

we deduce the following, of which the first is self-evident:

$$V_3 = 0;$$

Because	$V_3 = 0$,	$V_7 = 0$,	and $(v_5 = C_4)$;
,,	$(v_5 = C_4)$,	$V_9 = 0$,	and $(v_7 = C_6)$;
,,	$(v_7 = C_6)$,	$V_{13} = 0$,	and $(v_{11} = C_{10})$;
,,	$V_7 = 0$,	$V_{15} = 0$,	and $(v_{13} = C_{12})$;
,,	$V_9 = 0$,	$V_{19} = 0$,	and $(v_{17} = C_{16})$;
,,	$(v_{11} = C_{10})$,	$V_{21} = 0$,	and $(v_{19} = C_{18})$;
,,	$(v_{13} = C_{12})$,	$V_{25} = 0$,	and $(v_{23} = C_{22})$; &c.

Generally, $V_{6n+1} = 0 = V_{6n+3}$, for all values of n.

(C) Prop. $V_{6n} = 3n$; and $V_{6n+2} = 3n + 1$).

For if x be $6n$ or $6n + 2$, $V_{x+1} = 0$: also any letter a, occurring in Q_{x+1}, is found in $\frac{1}{2}x$ different triads, being combined in Q_{x+1} with x different letters. Let a be erased from Q_{x+1}: the remaining triads are Q_x, and the $\frac{1}{2}x$ duads that appear after the erasure are V_x, containing each of the x letters once. That this V_x is the least possible is plain from the following considerations.

Whether x is odd or even, any letter a that is found in Q_x m times, is therein combined with $2m$ different letters, and must appear in combination with $x-1-2m$ letters, i.e. $x-1-2m$, times, in V_x. When x is even this cannot be less than once.

Generally, any letter that is found in V_{2x}, is found in it an odd number of times; and any letter found in V_{2x+1} appears in it an even number of times.

The relation between Q_x and V_x may be expressed thus: Counting the duads, of which there are three in every triad, the following is always true, from the definitions of the terms:

$$3Q_x + V_x = D_x,$$

whether x is odd or even.

$$\text{If} \quad V_x = 0, \quad Q_x = \tfrac{1}{3}D_x = x \cdot \frac{x-1}{6}.$$

$$\text{If} \quad V_x = \tfrac{1}{2}x, \quad Q_x = \tfrac{1}{3}D_x - \frac{x}{6} = x \cdot \frac{x-2}{6};$$

$\therefore$ for the values of x, $x = 6n + 1$, $x = 6n + 3$, $x = 6n$, $x = 6n + 2$;

$$Q_x = x \cdot \frac{x - 2^{\cos^2 x \frac{1}{2}\pi}}{6}.$$

To find Q_x when $x = 6n - 1$ or $6n - 2$, which are the only cases not included in the above formula, is a more difficult matter; and the results about to be offered, although they will perhaps be assented to as certain, will yet be found deficient in mathematical rigour.

To consider Q_{2x+1} in general: all those letters that appear in it less than x times will be found in V_{2x+1}; and no one that appears x times in Q_{2x+1} can be exhibited in V_{2x+1}. Let there be y different letters found, and $x - y$ different letters not found, in V_x: $(x = 2n + 1)$. Let the y letters be $a, b, c. \ldots$ and the $x - y$ be $A, B, C. \ldots$

The triads in Q_x will in general be of the four forms

$$(ABC), \ (ADe), \ (Afg), \ (acd).$$

Of these forms let the numbers of triads in Q_x be, in that order,

$$I_x \quad M_x \quad R_x \quad T_x.$$

These numbers may vary much, as well as the number y, whilst the numbers x, Q_x, and V_x remain unchanged; so highly indeterminate is the structure of Q_x. But the following relations are true, of the duads employed.

(1) $$3I_x + M_x = D_{x-y},$$

for these duads are all exhausted in these two forms of triads;

(2) and $3T_x + R_x = D_y - V_x$.

(3) Also $3(I_x + M_x + R_x + T_x) = D_x - V_x = 3Q_x$,

(4) $\therefore\ 2(M_x + R_x) = D_x - (D_y + D_{x-y}) = y.(x-y)$.

If any relation can be assigned among these variables that shall lead to an equation determining V_x, a minimum, in terms of x; such an equation must be satisfied by $V_x = 0$, as well as by $V_x = fx$, fx being some simple function of the numbers x, 3, and 2; for V_x is not always reducible to 0: *e.g.* V_5. If now we suppose $y = V_x$, (4) − (2) gives

$$M_x - 3T_x = \tfrac{1}{2}y.(x-y) - D_y + V_x$$
$$= V_x.\frac{(x - V_x)}{2} - V_x.\frac{V_x - 1}{2} + V_x,$$

(5) or $$M_x - 3T_x = V_x.\left(\frac{x+3}{2} - V_x\right).$$

If either $M_x = 0 = T_x$ or $M_x = 3T_x$, we have an equation giving either $V_x = 0$ or $V_x = \frac{1}{2}(x+3)$. Since we know that the former is a minimum, we know that the conditions which lead to it are the conditions of a minimum in those cases in which $V_x = 0$ is possible; and this tempts us to conclude that we have hit upon the conditions of a minimum in general, and that the value of V_x is always $\frac{1}{2}(x+3)$, when it is not 0, (x being $2n+1$). Since this supposition is verified by trial in all cases that have been investigated, it is hoped that, until it is disproved by trial, the following theorem may be considered to have a good foundation,

$$(V_x =)\ V_{6n-1} = \frac{6n+2}{2} = 3n + 1.$$

It is however to be remarked, that the equation $V_x = \frac{1}{2}(x+3)$ is true only of x, such that V_x is excluded in the exhaustion of D_x by the formation of its duads into the triads of Q_x. If Q_x is of the form $Q_x = Q_{x'} + D_{x-x'}$, when $D_{x-x'}$ is added entire to $Q_{x'}$, as in the process above described, in which $Q_{15} = Q_7 + D_8$, D_8 being added entire; we have no right to expect a formula exhibiting V_x as a function of x. V_x in this case is in reality $V_{x'}$, being formed during the exhaustion of

D_x, and is a function, not of x, or of $x - x'$, but of x': hence the condition $M_x = 3T_x$ is to be understood of x' and Qx'.

$V_{6n'-1}$ is either V_{12n+5} or V_{12n+11}, as n' is odd or even.

(D) Prop. $V_{12n+5} = 6n + 4$.

Let Q_{6n+1} be formed: $V_{6n+1} = 0$. Let D_{6n+4} be written out with other $6n + 4$ letters a, b, c, &c., in $6n + 3$ columns, each containing all the $6n + 4$ letters. To these columns, in any order, let the letters $A, B, C. \ldots$, of which Q_{6n+1} is formed, be successively added as initials; then two of the $6n + 3$ columns will remain unemployed, and these $2.(3n + 2)$ duads are V_{12n+5} and $= 6n + 4$; and the $6n + 1$ columns of new triads being added to Q_{6n+1}, compose Q_{12n+5}. That V_{12n+5} is not $> 6n + 4$ is plain, because $V_{6n+1} = 0$; *i.e.* there are no duads made with ABC, &c. which are not employed in Q_{6n+1} which forms part of Q_{12n+5}; and each of $A, B, C \ldots$ is combined, in the formation of the added triads, with each of a, b, c, &c. That V_{12n+5} not $< 6n + 4$ it is difficult to prove, otherwise than by the arguments already adduced, and by induction from trial. In the manner above pointed out of forming Q_{12n+5}, $M_x = T_x = 0$, and in any case, by a little transformation, the condition $M_x = 3T_x > 0$, can be shewn to obtain; but if any one denies that these, along with $V_x = y$, are the conditions of a minimum, and asserts that V_x may be reduced by some other arrangement of triads below $6n + 4$, I confess myself unable to prove the negative. It is easy to shew that V_x, $(x = 2n + 1)$, can always be made equal to y; or, in other words, that V_x can always, by simple transpositions, be made to exhibit as many letters as duads. If it could be established, that *in every case* T_x can be made $= 0$, V_x remaining the same, by dispersion of the triads of the form (abc) into other forms, it could be easily shewn that M_x can be made $= 0$, and it would then follow of necessity by (5) that $V_x = \frac{1}{2}(x + 3)$.

Q_{17} is here formed, by adding to Q_7 7 columns of D_{10}.

ABC								
ADE	*BDF*	*CDG*						
AFG	*BEG*	*CEF*						
Ahi	*Bhk*	*Chl*	*Dhm*	*Ehn*	*Fho*	*Ghp*	*hq*	*hr*
Akq	*Bir*	*Cik*	*Dil*	*Eim*	*Fin*	*Gio*	*ip*	*iq*
Alp	*Blq*	*Cmq*	*Dkr*	*Ekl*	*Fkm*	*Gkn*	*ko*	*kp*
Amo	*Bmp*	*Cnp*	*Dnq*	*Eoq*	*Flr*	*Glm*	*ln*	*lo*
Anr	*Bno*	*Cor*	*Dop*	*Epr*	*Fpq*	*Gqr*	*mr*	*mn*

Here $M_{17} = I_{17} = 0$, $V_{17} = 10 = y_{17}$.

Q_{17} is here differently formed:

ABL									
AFP	*BIc*	*LIr*							
AIk	*BFg*	*LFq*							
AOe	*BPm*	*LOd*	*FOk*	*Pln*					
Acm	*BOr*	*LPe*	*FIm*	*POc*					
Adn	*Bhk*	*Lck*	*Fch*	*Pdr*	*Ioq*				
Ahg	*Bdq*	*Lng*	*Fed*	*Pkg*	*Idg*	*Omg*	*cgr*	*eqm*	
Aqr	*Ben*	*Lhm*	*Fnr*	*Pqh*	*Ieh*	*Onh*	*cqn*	*ekr*	*kdm.*

Here $M_{17} = 15 = 3T_{17}$

$I_{17} = 2 \quad R_{17} = 20$

$$\left\{\begin{array}{cc} \multicolumn{2}{c}{V_{17} = 10 = y_{17}} \\ gg & ge \\ ec & cd \\ dh & hr \\ rm & mn \\ nk & kq \end{array}\right\}$$

And again differently thus:

FIP							
FED	*IBL*	*PLE*					
FBg	*IEO*	*PDr*					
FAm	*IDn*	*POc*	*LOD*	*EBn*			
FOk	*IAk*	*PAn*	*LAr*	*EAh*	*DBh*	*OAq*	
FLq	*Icm*	*PBm*	*Lck*	*Ecg*	*DAg*	*OBr*	
Fch	*Ihg*	*Pkg*	*Lgn*	*Emq*	*Dcq*	*Omg*	*BAc*
Fnr	*Iqr*	*Pqh*	*Lhm*	*Ekr*	*Dkm*	*Onh*	*Bkq*

Here $I_{17} = 6, \; T_{17} = 0$

$M_{17} = 18,$

$R_{17} = 18,$

$$\left\{\begin{array}{cc} \multicolumn{2}{c}{V_{17} = 10 = y + 2} \\ gq & kh \\ gr & mn \\ cr & cn \\ mr & kn \\ hr & qn \end{array}\right\}$$

We conclude then, that $V_{12n+5} = V_{6(2n+1)-1} = 3(2n+1)+1 = 6n+4$, or that when n is odd, $V_{6n-1} = 3n+1$.

When n is even let $n = 2^m(2k+1)$, or let $2k+1$ be the greatest odd factor in n. In that case

(E) $$V_{6n-1} = 6k + 4.$$

The truth of this depends on the truth of the following proposition

$$V_{2x+1} = V_{4x+3}.$$

That this is true when $V_{2x+1} = 0$, we have already (A) seen; for in that case Q_{4x+3} is completed by the addition of D_{2x+2} entire to Q_{2x+1}; and $V_{4x+3} = 0$. The same is certainly true in all cases, if it be allowed that Q_{4x+3} contains all the triads in Q_{2x+1}; for whatever V_{2x+1} may be, it cannot by hypothesis be reduced by the aid of the $2x + 1$ letters $A, B, C. \ldots$ in Q_{2x+1}. If now D_{2x+2}, made with the $2x + 2$ letters $a, b, c. \ldots$ be placed at our disposal, it is absurd to require more than that D_{2x+2} shall be exhausted as well as all the duads possible with A and $abc. \ldots$, B and $abc. \ldots$, C and $abc. \ldots$, &c. Thus much we can always do; if then we add the D_{2x+2} new triads to Q_{2x+1} already formed, the sum will constitute Q_{4x+3}; and the duads V_{2x+1} are still what they were, and are all that remain to constitute V_{4x+3}. But as it is not in our power to prove that Q_{4x+3} must of necessity contain all the triads in Q_{2x+1} it may be denied, that as many duads must be excluded from the former as from the latter. Any person who thinks that $V_{23} < V_{11} < V_5$, or that $V_{35} < V_{17}$, is earnestly requested to try to shew it so to be; and till that is done, let the following proposition stand, as in all probability true;

$$V_{4x+3} = V_{2x+1}\,.\,(x = 2n + 1).$$

By the repeated application of this principle, it easily follows that

$$V_{6n-1} = V_{6\cdot 2^m\cdot(2k+1)-1} = V_{12k+5} = 6k + 4, \text{ (n being even)}:$$

and ($m \gtreqless 0$),

(E)′ $$V_{6n-1} = V_{6\cdot 2^m\cdot(2k+1)-1} = \qquad 6k + 4, \text{ n even or odd,}$$

and $= 2^m.(2k + 1)$.

(F) Prop. If n is odd, $V_{6n-2} = 3n - 1 + \frac{1}{2}(3n - 1)$.

Let $n = 2p + 1$; $6n - 2 = 12p + 4$. Let Q_{6p+2} be formed with the $6p + 2$ letters ABC.. Let D_{6p+2} made with other $6p + 2$ letters abc......be written out in $6p + 1$ columns, each containing all the letters abc...... Prefix $6p + 1$ letters ABC......, as before pointed out; and $(6p + 1)(3p + 1)$ triads are thus completed, which, being added to Q_{6p+2}, will form Q_{12p+4} or Q_{6n-2}. Let S be the one of the $6p + 2$ letters ABC....which is not employed as an initial letter; then the duads Sa, Sb, Sc, &c. are to be added to V_{6p+2} before given, and the sum forms $V_{12p+4} = V_{6p+2} + 6p + 2$; which means $V_{6n-2} = 3n - 1 + \frac{1}{2}(3n - 1)$; since we have already proved that $V_{6p+2} = 3p + 1 = \frac{1}{2}(3n - 1)$. Hence V_{6n-2} is not

greater than $3n - 1 + \frac{1}{2}(3n - 1)$. Neither is it less: for if it can be reduced, this can only be done by the aid of some letters repeated more than once in V_{6n-2}; the only repeated letter is S, for no letter is repeated in V_{6p+2}; and it is plainly impossible with the aid of S to form a new triad. This reasoning, although it may convince most readers, may yet be opposed by any person who refuses to concede that Q_{12p+4} a maximum can comprehend the triads Q_{6p+2}. Concluding, then, (F) that when n is odd, $V_{6n-2} = 3n - 1 + \frac{1}{2}(3n - 1)$, we proceed to show that, when n is even, and if $n = 2^m(2k + 1)$, or if $2k + 1$ be the greatest odd factor in n,

(G) $$V_{6n-2} = 3k + 1 + 3n - 1.$$

This is deduced immediately from the following, which we have now to prove:

$$V_{4n+2} = V_{2n} + n + 1.$$

When $V_{2n} = n$, or when $2n = 6n$ or $= 6n + 2$, this expresses what we have already established. For any value of V_{2n} the proof is easy. Let Q_{2n} be formed of the $2n$ letters ABC, &c. Let D_{2n+2}, made with the $2n + 2$ letters abc &c., be written out as before, in $2n + 1$ columns, each containing all the letters abc, &c. By prefixing the $2n$ letters ABC, &c., $2n$ columns of triads may be completed, and can be added to Q_{2n}. The ABC, &c., are thus each of them combined with each of abc, &c. One column of $n + 1$ duads remains of D_{2n+2} unemployed, and these being added to V_{2n}, compose, with it, V_{4n+2}: and $V_{4n+2} = V_{2n} + n + 1$. For, plainly, V_{4n+2} is not greater than this: neither is it less. For it contains no duads but of the forms (AB) and (cd), of which forms the first cannot be reduced in number, because V_{2n} is a minimum; nor can the second, since no letters are found in that form, except what must perforce be exhibited, viz, each of the letters abc....once.

$$\therefore\ V_{4n+2} = V_{2n} + n + 1.$$

Let $V_{2x} - x = \mu_{2x}$; then if $V_{2n} = n + \mu_{2n}$, $V_{4n+2} = n + n + 1 + \mu_{2n}$; but $V_{4n+2} - (2n + 1) = \mu_{4n+2}$; therefore the above may be expressed thus:

$$\mu_{4n+2} = \mu_{2n}.$$

Having already proved that (F)

$$\mu_{6n-2} = \tfrac{1}{2}(3n - 1) \text{ when } n \text{ is odd,}$$

by applying the above principle to $\mu_{6\cdot 2^m.(2k+1)-2}$, we easily find

$$\mu_{6\cdot 2^m.(2k+1)-2} = \mu_{6\cdot(2k+1)-2} = \frac{3.(2k+1)-1}{2} = 3k+1.$$

Hence, whether n is odd or even, if $n = 2^m.(2k+1)$, ($m \mathrel{\overline{\overline{>}}} 0$),

$$\mu_{6n-2} = 3k+1.$$

Also we have before established that, n being even or odd,

$$V_{6n-1} = 6k+4. \quad \text{(E}'\text{)}.$$

Q_x will therefore be found in all cases from the equation

$$3Q_x = D_x - V_x,$$

if for V_x we put $6k+4$ when $x = 6n-1$; $\frac{1}{2}x + 3k + 1$ when $x = 6n-2$; 0, when $x = 6n+1$ or $6n+3$; and $\frac{1}{2}x$ when $x = 6n$ or $6n+2$; n being $2^m(2k+1)$. The theorem in page 192 is therefore established, unless the proof be thought imperfect in the case of $x = 6n-1$.

It is worthy of remark that V_x is always either 0 or $\frac{1}{2}x$, or one of the two simple functions $\frac{1}{2}(x+3)$, $\frac{3}{2}.\frac{1}{2}x$; Q_x being formed by one complete operation; *i.e.* not being of the form $Q_{x'} + D_{x-x'}$.

If a general analytical expression be desired for Q_x, the following is one of the various forms in which it may be written:

$$Q_x = x.\frac{x - 2^{\cos^2(\frac{1}{4}x\pi)}}{6} -$$

$$\frac{t}{6}\{(t-1).\cos^2(\tfrac{1}{2}m\pi).(6k+4) + (t+1).\sin^2.(\tfrac{1}{2}m\pi).(3k+1);$$

where $x = 3m \pm t$, $t < 2$, and $2^r.(2k+1) = m + \sin^2(\frac{1}{2}m\pi)$; x, m, t, r, k being all integers $\mathrel{\overline{\overline{>}}} 0$.

Prob. Required the method of constructing Q_x, x being any number.

First let x be odd; it will either be $6n+1$ or $6n+3$, or $6n-1$. Exclude the latter, and let the two principles in page 196 be thus written.

If $V_{2n+1} = 0$, $V_{4n+3} = 0$ and $(v_{4n+1} = C_{4n})$

If $(v_{2n+1} = C_{2n})$, $V_{4n+1} = 0$ and $(v_{4n-1} = C_{4n-2})$.

If x is $4n+3$, it is necessary that Q_{2n+1} should be first formed, and all that is then required is the addition of D_{2n+2} entire, with the initial letters affixed. If x is $4n+1$, it is required that Q'_{2n+1} be first formed, and that v_{2n+1} shall be appended to it, as in page 195, a circle of $2n$; the obtaining

of which circle depends on our having previously formed Q_{2n+3}; the formation of which will again depend on the question whether $2n+3$ is $4n'+1$ or $4n'+3$. The following rule is safe and easy. In order to find x_1, such that Qx_1 is required to be formed, before Q_x can be completed; if x is $4n+3$, subtract 1 and divide by 2, giving $x_1 = 2n+1$; if x is $4n+1$, add 5 and divide by 2, giving $x_1 = 2n+3$. By the same rule $x_{\prime\prime}$ is to be found from x_1, &c. Thus in order to form Q_{609}, we have the following numbers: 609, $\overline{307}$, 153, $\overline{79}$, 39, 19, 9, $\overline{7}$, 3. The mark (–) over a number shows that we want, not, *e.g.* Q_{79}, but Q'_{77}, in order to complete it into Q_{153}; but it will be found that the easiest way to obtain Q'_{77} is to construct Q_{79}, and then to follow the directions given (*A*) and (*B*) p. 193. This mark is placed over a number when the next higher number is of the form $4n+1$.

When x is $6n-1$, it is $12n'+5$ or $12n'+11$. If the former, $Q_{6n'+1}$ must be formed, and then completed into $Q_{12n'+5}$ as page 199 (*D*) points out.

If x is $12n'+11$, Q_{12k+5} $\{n' = 2^m.(2k+1)\}$ must be formed; and this being Qx', all that is necessary is to add $D_{2x'+1}$, $D_{4x'+3}$, &c, in succession, till Qx is completed.

Q_{2x}, when $2x = 6n$ or $6n+2$, is obtained by erasing any letter from Q_{2x+1}. When $2x = 6n-2$, the mode of construction is apparent from p. 201, and what is said above of the formation of Q_{6n-1}.

Croft Rectory, near Warrington, Dec. 23, 1846.

ON SYMBOLICAL GEOMETRY.

By Sir William Hamilton.

[*Continued from* p. 133.]

Symbolical Expressions and Investigations of some Properties of Cyclic Cones, with reference to their Tangent Planes.

22. If the side b of the cyclic cone be conceived to approach to the side a, and ultimately to coincide with it, the first equation (152) will take this limiting form:

$$\frac{\mathrm{b}''}{\mathrm{a}} = \frac{\mathrm{a}}{\mathrm{a}'} \dots\dots\dots\dots\dots (155);$$

which expresses the known theorem that the side of contact a bisects the angle between the traces a′ and b″ of the

tangent plane on the two cyclic planes; bisecting also the vertically opposite angle between the traces $-a'$ and $-b''$, but being pependicular to the bisector of either of the two other angles, which are supplementary to the two already mentioned, namely the angle between the traces a' and $-b''$, and that between $-a'$ and b''. And if in like manner we conceive the side d to approach indefinitely to the side c, the plane of these two sides will tend to become another tangent plane to the cone; of which plane the traces c' and d'' on the two cyclic planes will satisfy an equation of the same form as that last written, namely the following, which is the limiting form of the third equation (152):

$$\frac{d''}{c} = \frac{c}{c'} \ldots\ldots\ldots\ldots\ldots\ldots (156).$$

At the same time, the two secant planes bc and da will tend to coalesce in one secant plane, containing the two sides of contact a and c, with which the two other sides b and d tend to coincide; so that the traces d' and a'' of the latter secant plane, on the two cyclic planes, will ultimately coincide with the traces b' and c'' of the former secant plane on the same two cyclic planes; and the equations (148) (153) become:

$$\frac{a'}{b'} = \frac{b'}{c'}; \quad \frac{b''}{c''} = \frac{c''}{d''} \ldots\ldots\ldots (157);$$

which express that the traces b' and c'' of the one remaining secant plane bisect respectively the angles between the pairs of traces, a', c', and b'', d'', of the two tangent planes, on the two cyclic planes. And the two remaining equations (152) concur in giving this other equation:

$$\frac{c''}{c} = \frac{a}{b'} \ldots\ldots\ldots\ldots\ldots (158):$$

expressing that the rotations in the secant plane from b' to a and from c to c'', that is to say from one trace to one side, and from the other side to the other trace, are equal in amount, and similarly directed; in such a manner that these two traces b' and c'', of the secant plane on the two cyclic planes, are equally inclined to the straight line which bisects the angle between these two sides a and c, along which the plane cuts the cone: all which agrees with the known properties of cones of the second degree.

23. The eight straight lines a, c, a', b', c', b'', c'', d'', being supposed to be equally long, the first of them, which has

been seen to coincide in direction with the bisector of the angle between the third and sixth, can differ only by a scalar (or real and numerical) coefficient from their symbolic sum; because the diagonals of a plane and equilateral quadrilateral figure (or rhombus) bisect the angles of that figure. We have therefore, by (155), and by the supposition of the equal lengths of the eight lines,

$$\mathrm{a}' + \mathrm{b}'' \parallel \mathrm{a}; \quad \text{or,} \quad \mathrm{a}' + \mathrm{b}'' = l\mathrm{a} \ldots\ldots (159),$$

l being a numerical coefficient, and the sign of parallelism being designed to include the case of coincidence.

In like manner, by (156), we have

$$\mathrm{d}'' + \mathrm{c}' \parallel \mathrm{c}; \quad \text{or,} \quad \mathrm{d}'' + \mathrm{c}' = l'\,\mathrm{c} \ldots\ldots (160),$$

l' being another scalar coefficient. Again, by (157),

$$\left.\begin{array}{ll} \mathrm{c}' + \mathrm{a}' \parallel \mathrm{b}'; & \mathrm{c}' + \mathrm{a}' = m\,\mathrm{b}'; \\ \mathrm{b}'' + \mathrm{d}'' \parallel \mathrm{c}''; & \mathrm{b}'' + \mathrm{d}'' = m'\,\mathrm{c}''; \end{array}\right\} \ldots (161),$$

m and m' being two other scalars. But, by (158),

$$\frac{\mathrm{c}''}{\mathrm{c}}\,\frac{\mathrm{b}'}{\mathrm{a}} = 1 \ldots\ldots\ldots\ldots\ldots (162);$$

therefore

$$\frac{\mathrm{b}'' + \mathrm{d}''}{\mathrm{d}'' + \mathrm{c}'}\,\frac{\mathrm{c}' + \mathrm{a}'}{\mathrm{a}' + \mathrm{b}''} = \frac{m'}{l'}\,\frac{m}{l} = V^{-1}0 \ldots\ldots (163);$$

this symbol $V^{-1}0$ denoting generally, in the present system, *any geometrical fraction of which the vector part is zero*, and therefore any positive or negative number (including zero). (Compare the definition and remarks in the 7th article).

By comparing this equation (163) with the first form (150), we see that the four straight lines,

$$-\mathrm{b}'', \mathrm{d}'', -\mathrm{c}', \mathrm{a}' \ldots\ldots\ldots\ldots (164),$$

which have been supposed to diverge from one common origin, namely the vertex of the cone, have their terminations on the circumference of one common circle. But these four lines, by supposition, are also equally long; they must therefore be four sides of a new cone, which is not only cyclic, as having a circular base, but is also a *cone of revolution.* The axis of revolution of this new cone is perpendicular to the plane of the circle in which the four lines (164) terminate; and this plane is parallel to the plane of the symbolic differences of those four lines, namely, the following,

$$\mathrm{d}'' + \mathrm{b}'', -\mathrm{c}' - \mathrm{d}'', \mathrm{a}' + \mathrm{c}', -\mathrm{b}'' - \mathrm{a}' \ldots (165);$$

but these have been seen to be parallel respectively to the four lines c″, c, b′, a, which are contained in the secant plane of the former cone; consequently the axis of revolution of the new cone is perpendicular to this secant plane. We arrive therefore, by this symbolical process, at a new proof of the known theorem, discovered by M. Chasles,* that two planes, touching a cyclic cone along any two sides, intersect the two cyclic planes in four right lines, which are sides of one common cone of revolution, whose axis of revolution is perpendicular to the plane of the two sides of contact of the former cone.

24. If we conceive the first and fourth of the sides (164) of the cone of revolution to tend to coincide with each other, then the fourth of the sides (165) of the plane quadrilateral inscribed in the circular base of that cone will tend to vanish; consequently the direction of this last mentioned side $-\mathrm{b}'' - \mathrm{a}'$, or the opposite direction of $\mathrm{a}' + \mathrm{b}''$, will become at last tangential to this circular base; and the plane of the two sides previously mentioned, namely $-\mathrm{b}''$ and a′, which plane has been seen to touch the cyclic cone along the side a, will become ultimately tangential also to the cone of revolution, touching it along the line a′, which becomes one trace of the second cyclic plane on the first cyclic plane; the opposite line, $-\mathrm{a}'$, being of course also situated in the intersection of those two planes, so that it may be regarded as the opposite trace of one cyclic plane on the other. Thus, at the limit here considered, the equation (155) and the second equation (157) are replaced by the equations

$$\frac{-\mathrm{a}'}{\mathrm{a}} = \frac{\mathrm{a}}{\mathrm{a}'}, \quad \frac{-\mathrm{a}'}{\mathrm{c}''} = \frac{\mathrm{c}''}{\mathrm{d}''} \ldots\ldots\ldots (166);$$

of which the first expresses that the side a is equally inclined to the two opposite traces, a′ and $-\mathrm{a}'$; while the numerical coefficient l vanishes, and the formula (163) is replaced by this other,

$$V.\frac{\mathrm{d}'' - \mathrm{a}'}{\mathrm{d}'' + \mathrm{c}'}\,\frac{\mathrm{c}' + \mathrm{a}'}{\mathrm{a}} = 0 \ldots\ldots\ldots\ldots (167).$$

We see also that the two rectangular but equally long lines a, a′, of which the former is a side of the cyclic cone,

* See the Translation of Two Geometrical Memoirs by M. Chasles, on the General Properties of Cones of the Second Degree, and on the Spherical Conics; which Translation was published, with an Appendix, by the Rev. Charles Graves, in Dublin, 1841.

while the latter is part of the line of intersection of the two cyclic planes of that cone, are such that their plane is a common tangent to both the cyclic cone and the cone of revolution; which latter cone has also, as sides of the same sheet with a′, the two other of the four lines (164), namely the lines − c′ and d″. Indeed, the formula (167) is sufficient to show, by comparison with the first formula (150), that if the three straight lines a′, d″, − c′ be still supposed to diverge from one common origin, the circle passing through the three points in which they terminate is touched, at the termination of the line a′, by a straight line parallel to the line a; and therefore that the cone of revolution, having these three equally long lines a′, − c′, d″ for sides of one common sheet, is touched along the side a′ by the plane which contains the two rectangular lines aa′; so that we may regard this formula (167) as containing the symbolical solution of the problem, to draw a tangent plane, along any proposed side, to the cone of revolution which passes through that side and through two other sides also given, and belonging to the same sheet as the former. Now if three such sides be connected by three planes, forming three faces of a triangular pyramid, inscribed in a single sheet of a cone of revolution, and having its vertex at the vertex of that cone, while the sheet is touched by a fourth plane along one edge of the pyramid, it follows from the most elementary principles of solid geometry, that the difference between the two exterior angles which the faces meeting at that edge make with the tangent plane to the cone is equal to the difference of the two interior angles which the same two faces make with the third face of the pyramid; the greater exterior angle being the one which is the more remote from the greater interior angle; as may be shown by conceiving three planes to pass through the three edges respectively, and through the axis of revolution of the cone. The same equality between the differences of these two pairs of angles between planes, will become still more evident if, without making use of any formula of spherical trigonometry, we consider a spherical triangle inscribed in a small circle on the sphere, which small circle is touched at one corner of the triangle by a great circle, while arcs are drawn to that and to the two other corners from a pole of the small circle; the only principles required being these: that the base angles of a spherical isosceles triangle are equal, and that the arcs from the pole of a small circle are all perpendicular to its perimeter. If then

we denote by the symbol $\angle\,(a, b, c)$ the acute or right or obtuse dihedral or spherical angle, at the edge b, between the planes ab and bc, in such a manner as to write, generally,

$$\begin{aligned}\angle\,(a, b, c) = \angle\,(c, b, a) = \angle\,(-a, b, -c) = \angle\,(a, -b, c)\\ = \pi - \angle\,(a, b, -c) \,\ldots\ldots\ldots\ldots\ldots\ldots (168);\end{aligned}$$

π being the symbol for two right angles, we shall have, in the present question, the equation

$$\angle(a', d'', -c') - \angle(a', -c', d'') = \angle(-a, a', -c') - \angle(a, a', d'') \,..(169);$$

and therefore, by subtracting both members from π,

$$\angle\,(a', d'', c') + \angle\,(a', c', d'') = \angle\,(-a, a', c') + \angle\,(a, a', d'') \,..(170).$$

We have also here the relation

$$\angle\,(c', a', d'') = \angle\,(a, a', c') + \angle\,(a, a', d'') \,..\,(171),$$

because the plane aa′ is intermediate between the planes a′c′ and a′ d″, or lies *within* the dihedral angle (c′, a′, d″) itself, and not within either of the two angles which are exterior and supplementary thereto; which again depends on the circumstance that both the cyclic planes are necessarily exterior to each sheet of the cyclic cone. Adding therefore the equations (170) and (171), member to member, and subtracting π on both sides of the result, we find for the *spherical excess* of the new triangular pyramid (a′, c′, d″), or for the excess of the sum of the mutual inclinations of its three faces a′c′, a′d″, c′d″, above two right angles, the expression:

$$\angle(a', d''\, c') + \angle(a', c', d'') + \angle(c', a', d'') - \pi = 2\,\angle\,(a, a', d'') \,..(172).$$

This spherical excess therefore remains unchanged, while the two lines c′, d″, move together on the two cyclic planes, in such a manner that their plane, always passing through the vertex of the cone, continues to touch that cyclic cone; a′ being still a line situated in the intersection of the two cyclic planes, and a being still a side of contact of the cone with a plane drawn through that intersection. And hence, or more immediately from the equation (170), the known property of a cyclic cone is proved anew, that the sum of the inclinations (suitably measured) of its variable tangent plane to its two fixed cyclic planes is constant.

(*To be continued.*)

ON THE DIFFERENTIAL EQUATIONS WHICH OCCUR IN DYNAMICAL PROBLEMS.

By Arthur Cayley.

Jacobi, in a very elaborate memoir, "Theoria novi multiplicatoris systemati æquationum differentialium vulgarium applicandi,"* has demonstrated a remarkable property of an extensive class of differential equations, namely, that when all the integrals of the system except a single one are known, the remaining integral can always be determined by a quadrature. Included in the class in question are, as Jacobi proceeds to shew, the differential equations corresponding to any dynamical problem in which neither the forces nor the equations of condition involve the velocities; *i.e.* in all ordinary dynamical problems when all the integrals but one are known the remaining integral can be determined by quadratures. In the case where the forces and equations of condition are likewise independent of the time, it is immediately seen that the system may be transformed into a system in which the number of equations is less by unity than in the original one, and which does not involve the time, which may afterwards be determined by a quadrature,† and Jacobi's theorem applying to this new system, he arrives at the proposition "In any dynamical problem where the forces and equations of condition contain only the coordinates of the different points of the system, when all the integrals but two are determined, the remaining integrals may be found by quadratures only. In the following paper, which contains the demonstrations of these propositions, the analysis employed by Jacobi has been considerably varied in the details, but the leading features of it are preserved.

§ 1. Let the variables $x, y, z \dots$ &c. be connected with the variables $u, v, w \dots$ by the same number of equations, so that the variables of each set may be considered as functions of those of the other set. And assume

$$dx\, dy \dots = \nabla\, du\, dv \dots$$

If from the functions which equated to zero express the relations between the two sets of variables we form two

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* *Crelle*, tom. XXVII. p. 199, and tom. XXIX. pp. 213 and 333. Compare also the memoir in *Liouville*, tom. x. p. 337.

† For, representing the velocities by $x', y' \dots$ the dynamical system takes the form $dt : dx : dy \dots : dx' : dy' \dots = 1 : x' : y' \dots : X : Y \dots$ and the system in question is simply $dx : dy \dots : dx' : dy' \dots = x' : y' \dots : X : Y \dots$

determinants, the former with the differential coefficients of these functions with respect to u, v,....and the latter with the differential coefficients of the same functions with respect to x, y,.....the quotient with its sign changed obtained by dividing the first of these determinants by the second is, as is well known, the value of the function ∇.

Putting for shortness

$$\frac{dx}{du}=\alpha, \quad \frac{dy}{du}=\beta.... \quad \frac{dx}{dv}=\alpha', \quad \frac{dy}{dv}=\beta'.... \text{ \&c.}$$

and $$\frac{du}{dx}=A, \quad \frac{du}{dy}=B.... \quad \frac{dv}{dx}=A', \quad \frac{dv}{dy}=B'....$$

∇ is the reciprocal of the determinant formed with $A, B...$; $A', B'......$, &c. Or it is the determinant formed with $\alpha, \beta, ...\alpha', \beta'....$, &c.

From the first of these forms, *i.e.* considering ∇ as a function of $A, B..$

$$\frac{d\nabla}{dA}=-\nabla\alpha, \quad \frac{d\nabla}{dB}=-\nabla\beta.... \quad \frac{d\nabla}{dA'}=-\nabla\alpha', \quad \frac{d\nabla}{dB'}=-\nabla\beta',$$

where the quantities $\alpha, \beta,..\alpha', \beta',..$and $A, B,..A', B',..$ may be interchanged provided $-\nabla$ be substituted for ∇. (Demonstrations of these formulæ or of some equivalent to them will be found in Jacobi's memoir "De determinantibus functionalibus," Crelle, t. XXII).

Hence

$$\frac{1}{\nabla}d\nabla+\alpha dA+\beta dB....+\alpha' dA'+\beta' dB'....=0.$$

or reducing by

$$\frac{dA}{dy}=\frac{dB}{dx}.... \quad \frac{dA'}{dy}=\frac{dB'}{dx}.... \text{ \&c.}$$

this becomes

$$\left.\begin{array}{l}\frac{1}{\nabla}d\nabla+\alpha\left(\frac{dA}{dx}dx+\frac{dB}{dx}dy+...\right)+\beta\left(\frac{dA}{dy}dx+\frac{dB}{dy}dy+...\right).. \\ \quad +\alpha\left(\frac{dA'}{dx}dx+\frac{dB'}{dx}dy+...\right)+\beta'\left(\frac{dA'}{dy}dx+\frac{dB'}{dy}dy+...\right)..\end{array}\right\}=0,$$

Or reducing

$$\frac{1}{\nabla}d\nabla+\left(\frac{dA}{du}+\frac{dA'}{dv}+..\right)dx+\left(\frac{dB}{du}+\frac{dB'}{dv}+..\right)dy+..=0;$$

whence separating the differentials and replacing $A, A',....$ $B, B',....$ by their values

$$\frac{1}{\nabla}\frac{d\nabla}{dx} + \frac{d}{du}\cdot\frac{du}{dx} + \frac{d}{dv}\cdot\frac{dv}{dx} + \ldots = 0,$$

$$\frac{1}{\nabla}\frac{d\nabla}{dy} + \frac{d}{du}\cdot\frac{du}{dy} + \frac{d}{dv}\cdot\frac{dv}{dy} + \ldots = 0,$$

$$\vdots$$

(in which $-\nabla, u, v \ldots x, y \ldots$ may be substituted for $\nabla, x, y \ldots u, v \ldots$).

§ 2. Let $X, Y \ldots$ be any functions of the variables $x, y \ldots$ and assume

$$U = X\frac{du}{dx} + Y\frac{du}{dy} + \ldots$$

$$V = X\frac{dv}{dx} + Y\frac{dv}{dy} + \ldots$$

$U, V, \ldots$ being expressed in terms of $u, v, \ldots$ Then

$$\frac{dU}{du} + \frac{dV}{dv} + \ldots$$

$$= X\left(\frac{d}{du}\cdot\frac{du}{dx} + \frac{d}{dv}\cdot\frac{dv}{dx} + \ldots\right) + Y\left(\frac{d}{du}\cdot\frac{du}{dy} + \frac{d}{dv}\cdot\frac{dv}{dy} + \ldots\right)\ldots$$

$$+ \left(\frac{dX}{du}\cdot\frac{du}{dx} + \frac{dX}{dv}\cdot\frac{dv}{dx} + \ldots\right) + \left(\frac{dY}{du}\cdot\frac{du}{dy} + \frac{dY}{dv}\cdot\frac{dv}{dy} + \ldots\right)\ldots$$

i.e. $\nabla\cdot\left(\frac{dU}{du} + \frac{dV}{dv} + \ldots\right)$

$$= -\left(X\frac{d\nabla}{dx} + Y\frac{d\nabla}{dy} + \ldots\right) + \nabla\cdot\left(\frac{dX}{dx} + \frac{dY}{dy} + \ldots\right).$$

Also, whatever be the value of M,

$$U\frac{dM\nabla}{du} + V\frac{dM\nabla}{dv} + \ldots = X\cdot\frac{dM\nabla}{dx} + Y\frac{dM\nabla}{dy} + \ldots$$

And from these two properties,

$$\frac{dM\nabla U}{du} + \frac{dM\nabla V}{du} + \ldots = \nabla\cdot\left(\frac{dMX}{dx} + \frac{dMY}{dy} + \ldots\right).$$

§ 3. Consider the system of differential equations

$$dx : dy : dz \ldots = X : Y : Z \ldots$$

(where, for greater clearness, an additional letter z has been introduced). From these we deduce the equivalent system

$$du : dv : dw \ldots = U : V : W \ldots$$

Suppose that u and v continue to represent arbitrary functions of x, y, z, but that the remaining function $w, \ldots$ is such as to

satisfy $W=0,\ldots$(so that $w,\ldots$ may be considered as the constants introduced by obtaining all the integrals but one of the system of differential equations in $x, y, z\ldots$), we have

$$\frac{dM\nabla U}{du}+\frac{dM\nabla V}{dv}=\nabla.\left(\frac{dMX}{dx}+\frac{dMY}{dy}+\frac{dMZ}{dz}+\ldots\right).$$

Also the only one of the transformed equations which remains to be integrated is

$$du : dv = U : V, \quad \text{or} \quad Vdu - Udv = 0,$$

(in which it is supposed that U and V are expressed by means of the other integrals in terms of u and v).

Suppose M can be so determined that

$$\frac{dMX}{dx}+\frac{dMY}{dy}+\frac{dMZ}{dz}+\ldots=0.$$

(M is what Jacobi terms the multiplier of the proposed system of differential equations). Then

$$\frac{dM\nabla U}{du}+\frac{dM\nabla V}{dv}=0,$$

or $M\nabla$ is the multiplier of $Vdu - Udv = 0$, so that

$$\int M\nabla\,(Vdu - Udv) = \text{const.}$$

Hence the theorem :—"Given a multiplier of the system of equations $dx : dy : dz \ldots = X : Y : Z \ldots$ (the meaning of the term being defined as above), then if all the integrals but one of this system are known, the remaining integral depends upon a quadrature."

Jacobi proceeds to discuss a variety of different systems of equations in which it is possible to determine the multiplier M. Among the most important of these may be considered the system corresponding to the general problem of Dynamics, which may be discussed under three different forms.

§ 4. Lagrange's first form.*

Let the *whole series* of coordinates, each of them multiplied by the square roots of the corresponding masses, be represented by $x, y \ldots$ and in the same way the whole series of forces, each of them multiplied by the square roots of the corresponding masses, by $P, Q \ldots$; then the equations of motion are

$$\frac{d^2x}{dt^2}=X,\quad \frac{d^2y}{dt^2}=Y\ldots$$

* I have slightly modified the form so as to avoid the introduction of the masses, and to allow x for instance to stand for any one of the coordinates of any of the points, instead of a coordinate parallel to a particular axis.

where
$$X = P + \lambda \frac{d\Theta}{dx} + \mu \frac{d\Phi}{dx} \ldots$$
$$Y = Q + \lambda \frac{d\Theta}{dy} + \mu \frac{d\Phi}{dy} \ldots$$
$$\vdots$$

where $\Theta = 0$, $\Phi = 0 \ldots\ldots$ are the equations of condition connecting the variables, and $\lambda, \mu \ldots\ldots$ coefficients to be determined by substituting the values of $\frac{d^2x}{dt^2}$ &c. in the equations $\frac{d^2\Theta}{dt^2} = 0$, $\frac{d^2\Phi}{dt^2} = 0$ &c. It is supposed that as well $P, Q \ldots$ as $\Theta, \Phi \ldots$ are independent of the velocities.

In order to reduce these to an analogous form to that previously employed, we have only to write
$$\frac{dx}{dt} = x', \quad \frac{dy}{dt} = y', \ldots$$
which gives
$$dt : dx : dy : dz \ldots : dx' : dy' : dz' \ldots$$
$$= 1 : x' : y' : z' \ldots : X : Y : Z \ldots$$

Supposing that M is independent of $x', y', z', \ldots$ the equation on which it depends becomes immediately
$$\delta M + M\left(\frac{dX}{dx'} + \frac{dY}{dy'} + \frac{dZ}{dz'} \ldots\right) = 0,$$
where for shortness
$$\delta = \frac{d}{dt} + x' \frac{d}{dx} + y' \frac{d}{dy} + z' \frac{d}{dz} + \ldots$$

To reduce this we must first determine the values of $\lambda, \mu \ldots$, and for this we have
$$\frac{d^2\Theta}{dt^2} = \delta^2\Theta + \frac{d\Theta}{dx} \cdot \frac{d^2x}{dt^2} + \frac{d\Theta}{dy} \cdot \frac{d^2y}{dt^2} + \ldots = 0, \text{ \&c.}$$

i.e. $$\delta^2\Theta + P\frac{d\Theta}{dx} + Q\frac{d\Theta}{dy} + \ldots + a\lambda + h\mu + g\nu \ldots = 0,$$
$$\delta^2\Phi + P\frac{d\Phi}{dx} + Q\frac{d\Phi}{dy} + \ldots + h\lambda + b\mu + f\nu \ldots = 0,$$
$$\delta^2\Psi + P\frac{d\Psi}{dx} + Q\frac{d\Psi}{dy} + \ldots + g\lambda + f\mu + c\nu \ldots = 0.$$
$$\vdots$$

where for greater clearness an additional letter of the series $\Theta, \Phi \ldots$ has been introduced, and where

$$a = \left(\frac{d\Theta}{dx}\right)^2 + \left(\frac{d\Phi}{dy}\right)^2 + \ldots .$$

$$b = \left(\frac{d\Phi}{dx}\right)^2 + \left(\frac{d\Phi}{dy}\right)^2 + \ldots .$$

$$\vdots$$

$$h = \left(\frac{d\Theta}{dx} \cdot \frac{d\Phi}{dx} + \frac{d\Theta}{dy} \cdot \frac{d\Phi}{dy}\right) + \ldots .$$

$$\vdots$$

Hence differentiating with respect to x',

$$2\delta \frac{d\Theta}{dx} + a \frac{d\lambda}{dx'} + h \frac{d\mu}{dx'} + g \frac{d\nu}{dx'} \ldots . = 0,$$

$$2\delta \frac{d\Phi}{dx} + h \frac{d\lambda}{dx'} + b \frac{d\mu}{dx'} + f \frac{d\nu}{dx'} \ldots . = 0,$$

$$2\delta \frac{d\Psi}{dx} + g \frac{d\lambda}{dx'} + f \frac{d\mu}{dx'} + c \frac{d\nu}{dx'} \ldots . = 0.$$

$$\vdots$$

Or representing by K the determinant formed with the quantities $a, h, g \ldots ; h, b, f \ldots g, f, c \ldots$ and by $A, H, G \ldots . H, B, F \ldots . G, F, C \ldots .$ the inverse system of coefficients, we have

$$2\left(A\delta \frac{d\Theta}{dx} + H\delta \frac{d\Phi}{dx} + G\delta \frac{d\Psi}{dx} \ldots\right) + K \frac{d\lambda}{dx'} = 0,$$

$$2\left(H\delta \frac{d\Theta}{dx} + B\delta \frac{d\Phi}{dx} + F\delta \frac{d\Psi}{dx} \ldots\right) + K \frac{d\mu}{dx'} = 0,$$

$$2\left(G\delta \frac{d\Theta}{dx} + F\delta \frac{d\Phi}{dx} + C\delta \frac{d\Psi}{dx} \ldots\right) + K \frac{d\nu}{dx'} = 0 ;$$

$$\vdots$$

whence multipling by $\frac{d\Theta}{dx}, \frac{d\Phi}{dx}, \frac{d\Psi}{dx} \ldots .$ and adding

$$A\delta \left(\frac{d\Theta}{dx}\right)^2 + B\delta \left(\frac{d\Phi}{dx}\right)^2 \ldots . + 2H\delta \frac{d\Theta}{dx} \cdot \frac{d\Phi}{dx} \ldots . + K \frac{dX}{dx'} = 0$$

and forming the similar equations with the remaining variables and adding

$$A\delta a + B\delta b + C\delta c \ldots + 2F\delta f + 2G\delta g + 2H\delta h + \ldots$$
$$+ K\left(\frac{dX}{dx'} + \frac{dY}{dy'} + \frac{dZ}{dz'} \ldots\right) = 0 ;$$

$$i.e. \qquad \delta K + K\left(\frac{dX}{dx'} + \frac{dY}{dy'} + \frac{dZ}{dz'} + \,..\right) = 0.$$

Or the equation in M reduces itself to

$$K\delta M - M\delta K = 0,$$

which is satisfied by $M = K$. It may be remarked that K reduces itself to the sum of the squares of the different functional determinants formed with the differential coefficients of Θ, Φ.... with respect to the different combinations of as many variables out of the series x, y....

§ 5. Lagrange's second form.

Here the equations of motion are assumed to be

$$\frac{d}{dt}\frac{dT}{dx'} - \frac{dT}{dx} - P = 0,$$

$$\frac{d}{dt}\frac{dT}{dy'} - \frac{dT}{dy} - Q = 0,$$

$$\frac{d}{dt}\frac{dT}{dz'} - \frac{dT}{dz} - R = 0.$$

$$\vdots$$

where $2T$ represents the vis viva of the system, $x, y, z,$.... are the independent variables on which the solution of the problem depends, and x', y', z'...their differential coefficients with respect to the time. It is assumed as before P, Q, R... do not contain x', y', z'..

Suppose these equations give

$$dt : dx : dy : dz \,..\, : dx' : dy' : dz' \,..$$
$$= 1 : x' : y' : z' \,..\, : X : Y : Z \,..$$

Then the equation which determines the multiplier M takes as before the form

$$\delta M + M\left(\frac{dX}{dx'} + \frac{dY}{dy'} + \frac{dZ}{dz'} + \,..\right) = 0.$$

To reduce this equation, substituting for T its value which is of the form

$$T = \tfrac{1}{2}\,.\,(ax'^2 + by'^2 + cz'^2 \,..\, + 2fy'z' + 2gz'x' + 2hx'y' \,..)$$

and putting for shortness

$$L = ax' + hy' + gz' \,....$$
$$M = hx' + by' + fz' \,....$$
$$N = gx' + fy' + cz' \,....$$
$$\vdots$$

The equations which determine $X, Y, Z. \ldots$ are

$$aX + hY + gZ \ldots + \delta L - \frac{dT}{dx} - P = 0,$$

$$hX + bY + fZ \ldots + \delta M - \frac{dT}{dy} - Q = 0,$$

$$gX + fY + cZ \ldots + \delta N - \frac{dT}{dz} - R = 0.$$

$$\vdots$$

Whence, differentiating with respect to x',

$$a\frac{dX}{dx'} + h\frac{dY}{dx'} + g\frac{dZ}{dx'} \ldots + \delta a \qquad = 0,$$

$$h\frac{dX}{dx'} + b\frac{dY}{dx'} + f\frac{dZ}{dz'} \ldots + \delta h + \frac{dM}{dx} - \frac{dL}{dy} = 0,$$

$$g\frac{dX}{dx'} + f\frac{dZ}{dx'} + c\frac{dZ}{dx'} \ldots + \delta g + \frac{dN}{dx} - \frac{dL}{dz} = 0.$$

$$\vdots$$

Or representing by K the determinant formed with $a, h, g \ldots$ $h, b, f, \ldots$ $g, f, c, \ldots$ and by $A, H, G, \ldots$ $H, B, F, \ldots$ $G, F, C \ldots$ the inverse system of coefficients, we have

$$K\frac{dX}{dx'} + A\delta a + H\delta h + G\delta g \ldots$$
$$+ \quad * \quad + H\left(\frac{dM}{dx} - \frac{dL}{dy}\right) + G\left(\frac{dN}{dx} - \frac{dL}{dz}\right) \ldots = 0,$$

and similarly

$$K\frac{dY}{dy'} + H\delta h + B\delta b + F\delta f \ldots$$
$$+ H\left(\frac{dL}{dy} - \frac{dM}{dx}\right) + \quad * \quad + F\left(\frac{dN}{dy} - \frac{dM}{dz}\right) \ldots = 0,$$

$$K\frac{dZ}{dz'} + G\delta g + F\delta f + C\delta c \ldots$$
$$+ G\left(\frac{dL}{dz} - \frac{dN}{dx}\right) + F\left(\frac{dM}{dz} - \frac{dN}{dy}\right) + \quad * \quad \ldots = 0.$$

$$\vdots$$

Whence, adding,

$$K\left(\frac{dX}{dx'} + \frac{dY}{dy'} + \frac{dZ}{dz'} \ldots\right)$$
$$+ A\delta a + B\delta b + C\delta c \ldots + 2F\delta f + 2G\delta g + 2H\delta h \ldots = 0;$$

and thus we have successively as before, though with symbols bearing an entirely different signification,

$$K\left(\frac{dX}{dx'}+\frac{dY}{dy'}+\frac{dZ}{dz'}+\,.\,.\right)+\delta K=0\,;$$

and thence $K\delta M - M\delta K = 0$, and $M = K$.

(The value of K *in this section* may I think be conveniently termed "the determinant of the vis viva," with respect to the variables x, y, z.. It may be remarked that "the determinant of the vis viva" with respect to any other system of variables $u, v, w,$..is $\nabla^2 K$, ∇ as before).

§ 6. Third form of the equations of motion.

Here writing

$$\frac{dT}{dx'}=\xi,\ \frac{dT}{dy'}=\eta\,.\,.$$

and taking $t, x, y,..\xi, \eta,..$ for the variables of the problem the equations of motion reduce themselves to

$$\begin{cases}\dfrac{d\xi}{dt}=-\dfrac{dT}{dx}+P, & \dfrac{dx}{dt}=\dfrac{dT}{d\xi},\\[2ex] \dfrac{d\eta}{dt}=-\dfrac{dT}{d\eta}+Q, & \dfrac{dy}{dt}=\dfrac{dT}{d\eta},\\ \quad\vdots & \quad\vdots\end{cases}$$

Or putting for shortness

$$\begin{cases}P-\dfrac{dT}{dx}=X, & \dfrac{dT}{d\xi}=\Xi,\\[2ex] Q-\dfrac{dT}{d\eta}=Y, & \dfrac{dT}{d\eta}=\mathrm{H},\\ \quad\vdots & \quad\vdots\end{cases}$$

they become

$$dt:dx:dy:dz\,.\,.:d\xi:d\eta:d\zeta\,.\,.$$
$$=1:\Xi:\mathrm{H}:\Omega\,.\,.:X:Y:Z\,.\,.$$

and writing the equation in M under the form

$$\delta M+M\,.\left(\frac{d\Xi}{dx}+\frac{d\mathrm{H}}{dy}+\,.\,.+\frac{dX}{d\xi}+\frac{dY}{d\eta}+\,.\,.\right)=0\,;$$

$$\left(\text{where } \delta=\frac{d}{dt}+\Xi\,\frac{d}{dx}+\mathrm{H}\,\frac{d}{dy}\,.\,.+X\,\frac{d}{d\xi}+Y\,\frac{d}{dy}+\,.\,.\right).$$

we see immediately that (P, Q. . being as before independent of the velocities, and consequently of ξ, η, ζ. .),

$$\frac{d\Xi}{dx} + \frac{dX}{d\xi} = 0, \quad \frac{dH}{dy} + \frac{dY}{d\eta} = 0, \text{ \&c.}$$

Hence $\delta M = 0$, which is satisfied by $M = 1$.

58, *Chancery Lane*, *Feb.* 6, 1847.

ON A MULTIPLE INTEGRAL CONNECTED WITH THE THEORY OF ATTRACTIONS.

By Arthur Cayley.

Mr. Boole has given for the integral with (n) variables

$$V = \int \frac{\phi\left(\frac{x^2}{f^2} + \frac{y^2}{g^2} + \,.\,.\right) dx\,dy\,.\,.}{[(a-x)^2 + (b-y)^2\,.\,.+ u^2]^{\frac{1}{2}n+q}} \;\ldots\ldots(1);$$

limits
$$\frac{x^2}{f^2} + \frac{y^2}{g^2} + \,.\,. \;\overline{\overline{<}}\; 1,$$

the following formula, or one which may readily be reduced to that form,*

$$V = \frac{fg\,.\,.\pi^{\frac{1}{2}n}}{\Gamma(\frac{1}{2}n + q)} \int_\eta^\infty \frac{Ss^{-q-1}ds}{\sqrt{\{(s+f^2)(s+g^2)\,.\,.}} \;\ldots.(2),$$

where
$$S = \frac{(1-\sigma)^{-q}}{\Gamma(-q)} \int_0^1 t^{-q-1}\phi\,\{\sigma + t(1-\sigma)\}\,dt \;\ldots.(3);$$

in which
$$\sigma = \frac{a^2}{f^2+s} + \frac{b^2}{g^2+s}\,.\,.+ \frac{u^2}{s} \;\ldots\ldots\ldots.(4),$$

and η is determined by

$$1 = \frac{a^2}{f^2+\eta} + \frac{b^2}{g^2+\eta}\,.\,.+ \frac{u^2}{\eta}.$$

Suppose $f = g = \ldots. = \infty$; also assume

$$\phi(\lambda) = \frac{1}{(f^2\lambda + v^2)^{\frac{1}{2}n+q'}} \;\ldots\ldots\ldots\ldots(5);$$

so that the integral becomes

$$U = \int \frac{dx\,dy\,.\,.}{(x^2+y^2\,.\,.+v^2)^{\frac{1}{2}n+q'}\,\{(x-a)^2 + .+ u^2\}^{\frac{1}{2}n+q}} \;\ldots(6),$$

the limits for each variable being $-\infty$, ∞.

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* See Note at the end of this paper.

Now, writing $f^2 s$ for s and $f^2\eta$ for η, the new value of η reduces itself to zero, and

$$U = \frac{f^{-2q}\,\pi^{\frac{1}{2}n}}{\Gamma(\frac{1}{2}n+q)} \int_0^\infty \frac{S\,ds}{(1+s)^{\frac{1}{2}n}}.$$

Also $\sigma = 0$; but $\qquad f^2\sigma = \dfrac{r^2}{1+s} + \dfrac{u^2}{s},$

where $r^2 = a^2 + b^2 + ..$ whence also putting $\dfrac{t}{f^2}$ for t, $\phi\{\sigma + t(1-\sigma)\}$ becomes

$$\frac{1}{\{f^2\sigma + t(1-\sigma) + v^2\}^{\frac{1}{2}n+q'}};$$

i.e. $$\frac{1}{(t+A)^{\frac{1}{2}n+q'}},$$

if for a moment $\qquad A = \dfrac{r^2}{1+s} + \dfrac{u^2}{s} + v^2.$

Hence
$$S = \frac{f^{2q}}{\Gamma(-q)} \int_0^\infty \frac{t^{-q-1}\,dt}{(t+A)^{\frac{1}{2}n+q'}}$$
$$= \frac{\Gamma(\frac{1}{2}n+q+q')}{\Gamma(\frac{1}{2}n+q')} \frac{f^{2q}}{A^{\frac{1}{2}n+q+q'}}.$$

Or substituting in U, and replacing A by its value,

$$U = \frac{\pi^{\frac{1}{2}n}\,\Gamma(\frac{1}{2}n+q+q')}{\Gamma(\frac{1}{2}n+q)\,\Gamma(\frac{1}{2}n+q')} \int_0^\infty \frac{s^{-q-1}\,ds}{(1+s)^{\frac{1}{2}n}\left(\dfrac{r^2}{1+s} + \dfrac{u^2}{s} + v^2\right)^{\frac{1}{2}n+q+q'}};$$

or, what comes to the same,

$$U = \frac{\pi^{\frac{1}{2}n}\,\Gamma(\frac{1}{2}n+q+q')}{\Gamma(\frac{1}{2}n+q)\,\Gamma(\frac{1}{2}n+q')} \int_0^\infty \frac{s^{\frac{1}{2}n+q'-1}(1+s)^{q+q'}\,ds}{(v^2s^2+js+u^2)^{\frac{1}{2}n+q+q'}} \;\ldots (7),$$

where $\qquad j = u^2 + v^2 + r^2.$

The only practicable case is of that $q' = -q$, for which

$$U = \frac{\pi^{\frac{1}{2}n}\,\Gamma(\frac{1}{2}n)}{\Gamma(\frac{1}{2}n+q)\,\Gamma(\frac{1}{2}n-q)} \int_0^\infty \frac{s^{\frac{1}{2}n-q-1}\,ds}{(v^2s^2+js+u^2)^{\frac{1}{2}n}} \;\ldots.(8).$$

Consider the more general expression

$$\Theta = \int_0^\infty s^{-q-1}\,\phi\left(\frac{v^2s^2+js+u^2}{s}\right) ds. \;\ldots\ldots(9).$$

By writing $\qquad 2v\sqrt{s} = \sqrt{(s'+4uv)} \pm \sqrt{s'},$

the upper sign from $s=\infty$ to $s=\frac{u}{v}$, and the lower one from $s=\frac{u}{v}$ to $s=0$, it is easy to derive

$$\Theta=(2v)^{2q}\int_0^\infty \frac{\{\sqrt{(s+4uv)}+\sqrt{s}\}^{-2q}+\{\sqrt{(s+4uv)}-\sqrt{s}\}^{-2q}}{\sqrt{s}\ \sqrt{(s+4uv)}}\,\phi\,(s+j+2uv)\,ds \quad \ldots\ldots (10).$$

Now, by a formula which will presently be demonstrated,

$$\int_0^\infty \frac{\{\sqrt{(s+4uv)}+\sqrt{s}\}^{-2q}+\{\sqrt{(s+4uv)}-\sqrt{s}\}^{-2q}}{\sqrt{s}\ \sqrt{(s+4uv)}}\,e^{-\theta s}ds$$

$$=\frac{2\sqrt{\pi}}{\Gamma(\frac{1}{2}-q)}\,\theta^{-q}\int_0^\infty s^{-\frac{1}{2}-q}\,(s+4uv)^{-\frac{1}{2}-q}\,e^{-\theta s}ds \quad \ldots\ldots (11);$$

whence
$$\int_0^\infty \frac{\{\sqrt{(s+4uv)}+\sqrt{s}\}^{-2q}+\{\sqrt{(s+4uv)}-\sqrt{s}\}^{-2q}}{\sqrt{s}\ \sqrt{(s+4uv)}}\,Fs.ds$$

$$=\frac{2\sqrt{\pi}}{\Gamma(\frac{1}{2}-q)}\int_0^\infty s^{-\frac{1}{2}-q}\,(s+4uv)^{-\frac{1}{2}-q}\left(-\frac{d}{ds}\right)^{-q}Fs.ds \quad \ldots\ldots (12).$$

So that by merely changing the function

$$\Theta=\frac{2^{2q+1}v^{2q}\sqrt{\pi}}{\Gamma(\frac{1}{2}-q)}\int_0^\infty s^{-\frac{1}{2}-q}\,(s+4uv)^{-\frac{1}{2}-q}\left(-\frac{d}{ds}\right)^{-q}\phi\,(s+j+2uv)\,ds \quad \ldots\ldots (13);$$

and thence in the particular case in question

$$U=\frac{2^{2q+1}v^{2q}\pi^{\frac{1}{2}(n+1)}}{\Gamma(\frac{1}{2}-q)\,\Gamma(\frac{1}{2}n-q)}\int_0^\infty s^{-\frac{1}{2}-q}\,(s+4uv)^{-\frac{1}{2}-q}\,(s+j+2uv)^{-\frac{1}{2}n+q}\,ds \quad \ldots\ldots (14),$$

by means of the formula

$$\left(-\frac{d}{ds}\right)^{-q}(s+a)^{-\frac{1}{2}n}=\frac{\Gamma(\frac{1}{2}n+q)}{\Gamma(\frac{1}{2}n)}\,(s+a)^{-\frac{1}{2}n+q}.$$

But as there may be some doubt about this formula, which is not exactly equivalent either to Liouville's or Peacock's expression for the general differential coefficient of a power, it is worth while to remark that, by first transforming the $\frac{1}{2}n^{\text{th}}$ power into an exponential, and then reducing as above, (thus avoiding the general differentiation), we should have obtained

$$U=\frac{2^{2q+1}v^{2q}\pi^{\frac{1}{2}(n+1)}}{\Gamma(\frac{1}{2}-q)\,\Gamma(\frac{1}{2}n+q)\,\Gamma(\frac{1}{2}n-q)}$$

$$\int_0^\infty d\theta\int_0^\infty ds\ \theta^{\frac{1}{2}n-q-1}\,e^{-\theta(s+j+2uv)}\,s^{-\frac{1}{2}-q}\,(s+4uv)^{-\frac{1}{2}-q}\,e^{-\theta s},$$

which reduces itself to the equation (14) by simply perform-

ing the integration with respect to θ; thus establishing the formula beyond doubt.* The integral may evidently be effected in finite terms when either q or $q-\frac{1}{2}$ is integral. Thus for instance in the simplest case of all, or when $q=-\frac{1}{2}$,

$$U=\frac{\pi^{\frac{1}{2}(n-1)}}{v\Gamma\frac{1}{2}(n+1)}\frac{1}{(j+2uv)^{\frac{1}{2}(n-1)}}$$

$$=\int_{-\infty}^{\infty}\frac{dx\,dy\ldots\ldots}{(x^2+y^2\ldots\ldots+v^2)^{\frac{1}{2}(n+1)}\left\{(x-a)^2+\ldots\ldots u^2\right\}^{\frac{1}{2}(n-1)}}.$$

A formula of which several demonstrations have already been given in the *Journal*.

The following is a demonstration, though an indirect one, of the formula (11): in the first place

$$\int_0^{\infty}\frac{\{\sqrt{(s+4uv)}+\sqrt{s}\}^{-2q}+\{\sqrt{(s+4uv)}-\sqrt{s})^{-2q}}{\sqrt{s}\,\sqrt{(s+4uv)}}e^{-\theta s}ds$$

$$=\frac{2\Gamma(\frac{1}{2}-q)\,\theta^q e^{2uv\theta}}{\sqrt{\pi}\,(4uv)^{2q}}\int_0^{\infty}(4u^2v^2+x^2)^{q-\frac{1}{2}}\,e^{i\theta x}dx\ldots.(16),$$

(where as usual $i=\sqrt{-1}$) to prove this, we have

$$\int_{-\infty}^{\infty}(4u^2v^2+x^2)^{q-\frac{1}{2}}e^{i\theta x}dx=\frac{1}{\Gamma(\frac{1}{2}-q)}\int_{-\infty}^{\infty}dx\int_0^{\infty}dt\,t^{-\frac{1}{2}-q}e^{-t(4u^2v^2+x^2)+i\theta x}$$

$$=\frac{\sqrt{\pi}}{\Gamma(\frac{1}{2}-q)}\int_0^{\infty}dt\,t^{-1-q}e^{-4u^2v^2t-\frac{\theta^2}{4t}}.$$

Or, putting $4uv\,\sqrt{t}=\sqrt{(s+4uv)}\pm\sqrt{s}$ (which is a transformation already employed in the present paper), the formula required follows immediately.

Now, by a formula due to M. Catalan, but first rigorously demonstrated by M. Serret,

$$\int_0^{\infty}\frac{\cos ax\,dx}{(1+x^2)^n}=\frac{\pi}{(\Gamma n)^2}\int_0^{\infty}e^{-(\alpha+2z)}\,(z+a)^{n-1}z^{n-1}dz,$$

(*Liouville*, tom. VIII., p. 1), and by a slight modification in the form of this equation

$$\int_{-\infty}^{\infty}(4u^2v^2+x^2)^{q-\frac{1}{2}}e^{i\theta x}\,dx=\frac{\pi e^{-2uv\theta}\,(4uv)^{2q}}{\theta^{2q}\Gamma^2(\frac{1}{2}-q)}\int_0^{\infty}s^{-q-\frac{1}{2}}(s+4uv)^{-q-\frac{1}{2}}e^{-\theta s}ds,$$

which, compared with (16), gives the required equation.

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* A paper by M. Schlömilch "Note sur la Variation des Constantes Arbitraires d'une Integrale definie," *Crelle*, tom. XXXIII. p. 268-280, will be found to contain formulæ analogous to some of the preceding ones.

NOTE.—One of the intermediate formulæ of Mr. Boole (*Irish Transactions*, tom. XXI.) may be written as follows:

$$S = \frac{1}{\pi} \int_0^1 da \int_0^\infty dv\, v^q \cos\left[(a - \sigma)\, v + \tfrac{1}{2} q\pi\right] \phi a.$$

Or what comes to the same thing, putting $i = \sqrt{-1}$, and rejecting the impossible part of the integral

$$S = \frac{1}{\pi} e^{\frac{1}{2}q\pi i} \int_0^1 da \int_0^\infty dv\, v^q\, e^{2v(a-\sigma)}\, \phi a,$$

$$\text{i.e. } S = \int_0^1 I\phi a da, \quad I = \frac{1}{\pi} e^{\frac{1}{2}q\pi i} \int_0^\infty dv\, v^q e^{iv(a-\sigma)}.$$

Now $(a - \sigma)$ being positive

$$I = \frac{1}{\pi} e^{\frac{1}{2}q\pi i}\, \Gamma(q + 1)\, e^{\frac{1}{2}(q+1)\pi i} (a - \sigma)^{-q-1};$$

i.e. $I = \frac{1}{\pi} e^{(q+\frac{1}{2})\pi i}\, \Gamma(q+1)\, (a-\sigma)^{-q-1}$, or relating the real part only

$$I = -\frac{1}{\pi} \sin q\pi\, \Gamma(q + 1)\, (a - \sigma)^{-q-1}; \text{ i.e. } I = \frac{1}{\Gamma(-q)} (a - \sigma)^{-q-1}.$$

But $(a - \sigma)$ being negative

$$I = \frac{1}{\pi} e^{\frac{1}{2}q\pi i}\, \Gamma(q + 1)\, e^{-\frac{1}{2}(q+1)\pi i} (\sigma - a)^{-q-1};$$

i.e. $I = \frac{1}{\pi} e^{-\frac{1}{2}\pi i}\, \Gamma(q + 1)\, (\sigma - a)^{-q-1}$, or retaining the real part only $I = 0$.

Hence $$S = \frac{1}{\Gamma(-q)} \int_\sigma^1 (a - \sigma)^{-q-1}\, \phi a da.$$

Or putting $a = \sigma + t(1 - \sigma)$, or $a - \sigma = t(1 - \sigma)$,

$$S = \frac{(1 - \sigma)^{-q}}{\Gamma(-q)} \int_0^1 t^{-q-1}\, \phi\left[\sigma + t(1 - \sigma)\right] dt,$$

the expression in the text. Mr. Boole's final value is

$$S = \left(-\frac{d}{d\sigma}\right)^q \phi(\sigma),$$

which though simpler appears to me to be in some respects less convenient.

ON THE EXISTENCE OF ROOTS OF ALGEBRAICAL EQUATIONS.

By the Rev. Harvey Goodwin, M.A., Caius College.

1. In a memoir printed in the *Cambridge Philosophical Transactions* (Vol. VIII. Part iii.), I have considered the roots of equations in the following manner. If $f(x) = 0$ represent any algebraical equation of n dimensions, the equation

$$z = f(x + y\sqrt{-1}) \dots\dots\dots\dots (1),$$

restricted to real values of z, will represent a curve of double curvature, the points of intersection of which with the plane of xy will determine by their distances from the origin the roots of the equation $f(x) = 0$. And I have proved that the curve consists of n infinite branches, which are continuous from $+\infty$ to $-\infty$, and therefore must cross the plane of xy in n points, and therefore determine n roots.

The mode of investigation adopted in the paper alluded to is applicable to prove the existence of the roots of equations without reference to geometrical considerations, and may be even extended without difficulty to a case discussed by Cauchy (*Exercises*, vol. IV.), namely, that in which the coefficients of the equation are imaginary quantities.

2. I must first explain the method which I adopt for representing the equation (1) when z is restricted to real values. We have

$$z = f(x + y\sqrt{-1})$$

$$= e^{y\sqrt{-1}\frac{d}{dx}} f(x) = \left(\cos y\frac{d}{dx} + \sqrt{-1}\sin y\frac{d}{dx}\right) f(x);$$

which, supposing the coefficients of $f(x)$ to be real, resolves itself into these two equations,

$$z = \left(\cos y\frac{d}{dx}\right) f(x) \dots\dots\dots\dots (2),$$

$$0 = \left(\sin y\frac{d}{dx}\right) f(x) \dots\dots\dots\dots (3).$$

The expressions $\cos y\dfrac{d}{dx}$ and $\sin y\dfrac{d}{dx}$ are to be supposed expanded in powers of $y\dfrac{d}{dx}$, and when the differentiations indicated are performed, the equations (2) and (3) will consist of only a finite number of terms.

3. If $P = \left(\cos\, y\dfrac{d}{dx}\right) f(x)$ and $Q = \left(\sin\, y\dfrac{d}{dx}\right) f(x)$, it is easy to see that

$$\frac{dP}{dx} = \left(\cos\, y\frac{d}{dx}\right) f'(x), \qquad \frac{dP}{dy} = -\left(\sin\, y\frac{d}{dx}\right) f'(x),$$

$$\frac{dQ}{dx} = \left(\sin\, y\frac{d}{dx}\right) f'(x), \qquad \frac{dQ}{dy} = \left(\cos\, y\frac{d}{dx}\right) f'(x),$$

$$\frac{d^2P}{dx^2} = \left(\cos\, y\frac{d}{dx}\right) f''(x), \qquad \frac{d^2P}{dx\,dy} = -\left(\sin\, y\frac{d}{dx}\right) f''(x),$$

$$\frac{d^2P}{dy^2} = -\left(\cos\, y\frac{d}{dx}\right) f''(x),$$

$$\frac{d^2Q}{dx^2} = \left(\sin\, y\frac{d}{dx}\right) f''(x), \qquad \frac{d^2Q}{dx\,dy} = \left(\cos\, y\frac{d}{dx}\right) f''(x),$$

$$\frac{d^2Q}{dy^2} = -\left(\sin\, y\frac{d}{dx}\right) f''(x),$$

&c. &c.

$$\text{therefore } \frac{dP}{dx} = \frac{dQ}{dy},$$

$$\frac{dQ}{dx} = -\frac{dP}{dy},$$

$$\frac{d^2P}{dx^2} = \frac{d^2Q}{dx\,dy} = -\frac{d^2P}{dy^2},$$

$$\frac{d^2Q}{dx^2} = -\frac{d^2P}{dx\,dy} = -\frac{d^2Q}{dy^2},$$

&c. &c.

4. Theorem:—*If $f(x) = x^n + p_1 x^{n-1} + p_2 x^{n-2} + \ldots\ldots + p_n$, where $p_1, p_2 \ldots p_n$, are either real or imaginary, then if x be allowed to assume all values real or imaginary, subject to the condition that $f(x)$ is real, $f(x)$ does not admit of a maximum or minimum value.*

5. For simplicity's sake suppose first the coefficients $p_1, p_2 \ldots$ to be real. Then putting for x, $x + y\sqrt{-1}$, the equation $z = f(x)$ resolves itself into these two:

$$z = \left(\cos\, y\frac{d}{dx}\right) f(x) = P,$$

$$0 = \left(\sin\, y\frac{d}{dx}\right) f(x) = Q.$$

For a maximum or minimum we must have $\delta z = 0$;

$$\left.\begin{aligned} &\text{therefore} && \frac{dP}{dx}\,\delta x + \frac{dP}{dy}\,\delta y = 0 \\ &\text{also} && \frac{dQ}{dx}\,\delta x + \frac{dQ}{dy}\,\delta y = 0 \end{aligned}\right\} \ldots\ldots\ldots.(4).$$

Multiplying these equations by $\dfrac{dP}{dx}$, $\dfrac{dQ}{dx}$ respectively, and observing the relations established in Art. 3, we have

$$\frac{dP^2}{dx^2} + \frac{dQ^2}{dx^2} = 0,$$

$$\left.\begin{aligned} &\text{therefore} && \frac{dP}{dx} = 0, \quad \frac{dQ}{dx} = 0 \\ &\text{therefore also} && \frac{dP}{dy} = 0, \quad \frac{dQ}{dy} = 0 \end{aligned}\right\} \ldots\ldots\ldots\ldots.(5).$$

In order to ascertain whether the values of x and y, given by these equations, correspond to a maximum or minimum value of z, the value of $\delta^2 z$ must be examined. We have

$$\left.\begin{aligned} 1.2.\delta^2 z &= \frac{d^2P}{dx^2}\,\delta x^2 + 2\,\frac{d^2P}{dx\,dy}\,\delta x\,\delta y + \frac{d^2P}{dy^2}\,\delta y^2 \\ \text{and}\quad 0 &= \frac{d^2Q}{dx^2}\,\delta x^2 + 2\,\frac{d^2Q}{dx\,dy}\,\delta x\,\delta y + \frac{d^2Q}{dy^2}\,\delta y^2 \end{aligned}\right\} ..(6),$$

(since the coefficients of the quantity $\delta^2 y$ are zero).

Let A and B be the values assumed by $\dfrac{d^2P}{dx^2}$ and $\dfrac{d^2Q}{dx^2}$, corresponding to the values of x and y given by equations (5). Then, by the relations established in Art 3,

$$1.2.\delta^2 z = A\delta x^2 - 2B\,\delta x\,\delta y - A\delta y^2,$$

$$0 = B\delta x^2 + 2A\,\delta x\,\delta y - B\delta y^2.$$

Let $\qquad \delta x = \varepsilon \cos \phi,$

$$\delta y = \varepsilon \sin \phi;$$

therefore $\quad 1.2\,\delta^2 z = \varepsilon^2 \{A \cos 2\phi - B \sin 2\phi\},$

$$0 = \varepsilon^2 \{B \cos 2\phi + A \sin 2\phi\},$$

which equations may be put under the form

$$1.2.\delta^2 z = \varepsilon^2 C \cos (2\phi + a),$$

$$0 = \sin (2\phi + a);$$

therefore $\qquad 2\phi + a = 0 \text{ or } \pi,$

and $\qquad 1.2\,\delta^2 z = \pm\,\varepsilon^2 C.$

Hence $\delta^2 z$ has two values, one positive and the other negative, and therefore the value of z corresponding to the values of x and y, supposed to be obtained, cannot be said to be either a maximum or a minimum.

6. But it is possible that $\delta^2 z$ may vanish; we shall therefore consider the general case in which $\delta^m z$ is the first of the series of quantities δz, $\delta^2 z$, $\delta^3 z$, &c., which does not vanish; and it is not difficult to see that in this case we have

$$1.2\ldots m\,\delta^m z = \left(\delta x\frac{d}{dx} + \delta y\frac{d}{dy}\right)^m P,$$

$$0 = \left(\delta x\frac{d}{dx} + \delta y\frac{d}{dy}\right)^m Q.$$

Let A, B be the the values assumed by $\dfrac{d^m P}{dx^m}$, $\dfrac{d^m Q}{dx^m}$, respectively; then, in consequence of the relations of Art. 3, we shall have

$$1.2\ldots m\delta^m z = A\delta x^m - mB\delta x^{m-1}\delta y - \frac{m.(m-1)}{1.2}A\delta x^{m-2}\delta y^2 + \ldots$$

$$0 = B\delta x^m + mA\delta x^{m-1}\delta y - \frac{m.(m-1)}{1.2}B\delta x^{m-2}\delta y^2 - \ldots$$

Let $\delta x = \varepsilon\cos\phi$, $\delta y = \varepsilon\sin\phi$; therefore

$$1.2\ldots m\delta^m z = \varepsilon^m\left\{A\left(\cos^m\phi - \frac{m.(m-1)}{1.2}\cos^{m-2}\phi\sin^2\phi + \ldots\right)\right.$$
$$\left. - B(m\cos^{m-1}\phi\sin\phi\ldots)\right\},$$

$$0 = \varepsilon^m\left\{B\left(\cos^m\phi - \frac{m.(m-1)}{1.2}\cos^{m-2}\phi\sin^2\phi + \ldots\right)\right.$$
$$\left. + A(m\cos^{m-1}\phi\sin\phi\ldots)\right\},$$

or
$$1.2\ldots m\delta^m z = \varepsilon^m\{A\cos m\phi - B\sin m\phi\},$$
$$0 = \varepsilon^m\{B\cos m\phi + B\sin m\phi\};$$

which may be put under the form

$$1.2\ldots m\delta^m z = \varepsilon^m C\cos(m\phi + a),$$
$$0 = \sin(m\phi + a);$$

therefore $m\phi + a = k\pi$, where k may have any one of the values $0, 1, 2\ldots\ldots m-1$, and

$$1.2\ldots m\delta^m z = (-1)^k\varepsilon^m C.$$

Therefore $\delta^m z$ has m values which are alternately positive and negative, and therefore z admits of no maximum or minimum value.

7. We must now consider the still more general case, in which one or more of the coefficients $p_1, p_2 \ldots\ldots p_n$ are imaginary.

Let
$$p_1 = r_1 (-1)^{\frac{a_1}{\pi}} = r_1 (\cos a_1 + \sqrt{-1} \sin a_1),$$

$$p_2 = r_2 (-1)^{\frac{a_2}{\pi}} = r_2 (\cos a_2 + \sqrt{-1}. \sin a_2),$$
&c. = &c.

therefore
$$\begin{aligned} f(x) &= x^n + p_1 x^{n-1} + p_2 x^{n-2} + \ldots\ldots \\ &= x^n + r_1 \cos a_1 x^{n-1} + r_2 \cos a_2 x^{n-2} + \ldots\ldots \\ &\quad + \sqrt{-1} \left\{ r_1 \sin a_2 x^{n-1} + r_2 \sin a_2 x^{n-2} + \ldots\ldots \right\} \\ &= M + N\sqrt{-1} \text{ suppose.} \end{aligned}$$

$$\begin{aligned} \therefore\ f(x + y\sqrt{-1}) &= \left(\cos y \frac{d}{dx} + \sqrt{-1} \sin y \frac{d}{dx}\right) M \\ &\quad + \left(\cos y \frac{d}{dx} + \sqrt{-1} \sin y \frac{d}{dx}\right) N\sqrt{-1} \\ &= \left(\cos y \frac{d}{dx}\right) M - \left(\sin y \frac{d}{dx}\right) N \\ &\quad + \sqrt{-1} \left\{ \left(\sin y \frac{d}{dx}\right) M + \left(\cos y \frac{d}{dx}\right) N \right\}. \end{aligned}$$

Hence if
$$f(x + y\sqrt{-1}) = P + Q\sqrt{-1},$$
we have
$$z = P = \left(\cos y \frac{d}{dx}\right) M - \left(\sin y \frac{d}{dx}\right) N,$$
$$0 = Q = \left(\sin y \frac{d}{dx}\right) M + \left(\cos y \frac{d}{dx}\right) N.$$

Differentiating
$$\frac{dP}{dx} = \left(\cos y \frac{d}{dx}\right) \frac{dM}{dx} - \left(\sin y \frac{d}{dx}\right) \frac{dN}{dx},$$
$$\frac{dP}{dy} = - \left(\sin y \frac{d}{dx}\right) \frac{dM}{dx} - \left(\cos y \frac{d}{dx}\right) \frac{dN}{dx},$$
$$\frac{dQ}{dx} = \left(\sin y \frac{d}{dx}\right) \frac{dM}{dx} + \left(\cos y \frac{d}{dx}\right) \frac{dN}{dx},$$
$$\frac{dQ}{dy} = \left(\cos y \frac{d}{dx}\right) \frac{dM}{dx} - \left(\sin y \frac{d}{dx}\right) \frac{dN}{dx},$$
$$\therefore\ \frac{dP}{dx} = \frac{dQ}{dy}, \quad \frac{dP}{dy} = - \frac{dQ}{dx};$$

and in like manner we should find that

$$\frac{d^2P}{dx^2} = \frac{d^2Q}{dx\,dy} = -\frac{d^2P}{dy^2},$$

$$\frac{d^2Q}{dx^2} = -\frac{d^2P}{dx\,dy} = -\frac{d^2Q}{dy^2},$$

and so on, as in Art. 3. Hence the relations between the differential coefficients of P and Q being the same as in the case of the quantities $p_1, p_2 \dots$ being real, the investigation given in that case is equally applicable to the more general one now under consideration.

Hence the theorem enunciated is true.

8. Since then $f(x)$ admits of no maximum or minimum value, we may give x a succession of values of the form $x + y\sqrt{-1}$, which shall cause $f(x)$ to assume all real values intermediate to $+\infty$ and $-\infty$. Let us now examine how many such sets of values can be found.

When x is very large, the equation

$$z = x^n + p_1 x^{n-1} + \dots$$

degenerates into the following,

$$z = x^n.$$

Let $$x = \rho\,(\cos\theta + \sqrt{-1}\sin n\theta),$$
therefore $$z = \rho^n(\cos n\theta + \sqrt{-1}\sin\theta);$$
which is equivalent to these two,

$$z = \rho^n \cos n\theta,$$
$$0 = \sin n\theta;$$

therefore $n\theta = k\pi$, where k may have any one of the values $0, 1, 2 \dots (2n-1)$, and

$$z = (-1)^k \rho^n.$$

Hence, when $k = 0, 2, 4 \dots$, z becomes $+\infty$ when ρ is indefinitely increased, and when $k = 1, 3, 5 \dots$, becomes $-\infty$; and therefore there are n series of values which may be assigned to x, which will make z vary continuously from $+\infty$ and $-\infty$, and therefore n values may be found which will make z vanish, that is, the equation $f(x) = 0$ has n roots.

Cambridge, Feb. 10, 1847.

ON THE FORCES EXPERIENCED BY SMALL SPHERES UNDER MAGNETIC INFLUENCE; AND ON SOME OF THE PHENOMENA PRESENTED BY DIAMAGNETIC SUBSTANCES.

By William Thomson.

The circumstance that a magnet* attracts small pieces of iron, is the phenomenon of magnetism which was first observed; and an analogous action, presented by rubbed amber, first drew attention to the phenomena of electricity. Now it has since been discovered that no mutual attraction or repulsion between two bodies can result from magnetism in one, unless the other be also magnetized, and that no electric force can exist unless each body be electrically excited. Hence it appears that the forces originally observed are the consequences of a temporary magnetic or electric state induced in a neutral body, when placed in the neighbourhood of a magnet or of an electrified body.

In the following paper the law of such phenomena with reference to magnetism† is considered. It is easily shown however that, by taking $i = 1$ in the formulæ obtained below, the corresponding results for small insulated conductors, electrified by influence, may be obtained, although the physical problems are entirely distinct.

1. We may commence by considering the case of a small sphere of soft iron, or of any other substance susceptible of magnetic induction; and it is easily shewn that the formula expressing the results may be applied to the case of a small cube by merely altering the value of a certain coefficient; and in general to the case of a small portion of matter of any form, such that in whatever way it be turned round the resultant axis of magnetization, for the whole mass, shall coincide with the direction of the magnetizing force.

2. It is well known that if a small homogeneous sphere

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* Originally a piece of magnetic iron ore or loadstone. The term may now be applied to any mass possessing permanent magnetism, and may even be extended to a galvanic wire of any form.

† This has not been made the subject of a special investigation by any writer, so far as I am aware, although the nature of the result, in the case of magnetism, appears to be entirely understood by Mr. Faraday. Thus, from § 2418 (quoted below, in the text,) of his *Experimentnl Researches*, we might infer that a small sphere or cube of soft iron would in some cases be "urged along, and in others obliquely or directly across the lines of magnetic force"; and that all the phenomena would resolve themselves into this, that such a portion of matter, when under magnetic action, tends to move from weaker to stronger places or points of force.

of soft iron, or of any other substance susceptible of magnetic induction, be placed in the neighbourhood of a magnet, it will become uniformily magnetized, throughout its mass, with an intensity numerically expressed by multiplying the magnetizing force, by a coefficient independent of the dimensions of the sphere. Thus if R denote the resultant force of the magnet, or the force that it would exert upon an imaginary unit of magnetism, at the position occupied by the sphere, of which we suppose the dimensions to be so small that R has sensibly the same value and direction throughout; and if κ be the intensity of the induced magnetism; we have

$$\kappa = \cancel{\frac{4\pi i}{3}}\, R\, \frac{3i}{4\pi} \ldots\ldots\ldots\ldots (1),$$

where i is a proper fraction (nearly equal to unity for soft iron) depending on the capacity of the substance for magnetic induction.

3. If the force R were rigorously constant in magnitude and direction throughout the whole space S occupied by the sphere, then there would be no resulting force tending to move the sphere; as, for example, we may conceive it to be, without committing an appreciable error, in the case of a ball of iron of any ordinary dimensions magnetized by the terrestrial force. In the investigation which follows we shall therefore have to consider the small variation of R through the space S but, although considering the effect of this small variation in causing a moving force upon the magnetized sphere, we may neglect the deviation from rigorous uniformity of magnetization which it will produce.

4. Let X, Y, Z be the components of R at the point (x, y, z), which may be taken as the centre of the small sphere. At any point $(x+f)$, $(y+g)$, $(z+h)$, in the sphere, we shall have, for the components of the resultant force due to the magnet,

$$X + \frac{dX}{dx} f + \frac{dX}{dy} g + \frac{dX}{dz} h,$$

$$Y + \frac{dY}{dx} f + \frac{dY}{dy} g + \frac{dY}{dz} h,$$

$$Z + \frac{dZ}{dx} f + \frac{dZ}{dy} g + \frac{dZ}{dz} h.$$

By considering the effects of these forces upon the elements (as for instance thin bars, in the direction of magnetization)

into which the magnetized sphere may be supposed to be divided, it is easily shewn, as has also been done by Poisson, that the components of the resulting force on the sphere are given by the equations

$$F = \frac{dX}{dx}.\kappa\sigma.l + \frac{dX}{dy}.\kappa\sigma.m + \frac{dX}{dz}.\kappa\sigma.n,$$

$$G = \frac{dY}{dx}.\kappa\sigma.l + \frac{dY}{dy}.\kappa\sigma.m + \frac{dY}{dz}.\kappa\sigma.n,$$

$$H = \frac{dZ}{dx}.\kappa\sigma.l + \frac{dZ}{dy}.\kappa\sigma.m + \frac{dZ}{dz}.\kappa\sigma.n,$$

where σ is the volume of the sphere, and l, m, n the cosines of the angles made by the direction of magnetization with the axes. Now since this direction is that of the force R, we have

$$l = \frac{X}{R}, \quad m = \frac{Y}{R}, \quad n = \frac{Z}{R}.$$

Hence, since $\kappa = \frac{4\pi}{3} i.R$, we have

$$\left.\begin{aligned} F &= \frac{4\pi i}{3}\sigma\left(X\frac{dX}{dx} + Y\frac{dX}{dy} + Z\frac{dX}{dz}\right) \\ G &= \frac{4\pi i}{3}\sigma\left(X\frac{dY}{dx} + Y\frac{dY}{dy} + Z\frac{dY}{dz}\right) \\ H &= \frac{4\pi i}{3}\sigma\left(X\frac{dZ}{dx} + Y\frac{dZ}{dy} + Z\frac{dZ}{dz}\right) \end{aligned}\right\} \cdots\cdots (2).$$

5. Now if R be due to any magnet, or to a closed galvanic current, $X\,dx + Y\,dy + Z\,dz$ is necessarily a complete differential, and therefore we have

$$\frac{dY}{dz} = \frac{dZ}{dy}, \quad \frac{dZ}{dx} = \frac{dX}{dz}, \quad \frac{dX}{dy} = \frac{dY}{dx} \cdots\cdots (3).$$

Modifying the second members of (2) by means of these equations, we find

$$\left.\begin{aligned} F &= \frac{3\cancel{4\pi}i}{4\pi}\sigma\left(X\frac{dX}{dx} + Y\frac{dY}{dx} + Z\frac{dZ}{dx}\right) = \frac{4\pi i}{3}\sigma.R\frac{dR}{dx} \\ G &= \frac{3\cancel{4\pi}i}{4\pi}\sigma\left(X\frac{dX}{dy} + Y\frac{dY}{dy} + Z\frac{dZ}{dy}\right) = \frac{4\pi i}{3}\sigma.R\frac{dR}{dy} \\ H &= \frac{3\cancel{4\pi}i}{4\pi}\sigma\left(X\frac{dX}{dz} + Y\frac{dY}{dz} + Z\frac{dZ}{dz}\right) = \frac{4\pi i}{3}\sigma.R\frac{dR}{dz} \end{aligned}\right\} \cdots (4).$$

From these we deduce

$$Fdx + Gdy + Hdz = \frac{\overset{3i}{\cancel{4\pi i}}}{4\pi}\,\sigma\,.\,RdR = d\left(\frac{3\pi i}{8}\,\sigma\,.\,R^2\right)\ldots(5),$$

which expresses fully the result of equations (4).

6. The interpretation of this result shows that a sphere of soft iron is urged in the direction in which the magnetizing force increases most rapidly; the components of the force in different directions being expressible by the differential coefficients of the function $\frac{2\pi i}{3}\,\sigma R^2$. Thus in some cases it may actually be urged across the direction of the magnetizing force. For instance, if a ball of soft iron be placed symmetrically with respect to the two poles of a horse-shoe magnet, and at some distance from the line joining them, it will be urged towards this line in a direction perpendicular to it, although the magnetizing force is parallel to it; or if the magnetizing force be due to a straight galvanic wire, a ball of soft iron will be *attracted* towards the wire, although the force on an imaginary "magnetic point" is perpendicular to a plane through it and the wire.

7. The positions of equilibrium of a small sphere acted upon by the magnetic forces alone, will be points in the neighbourhood of which R^2 is stationary in value, or points where $d\,(R^2) = 0$. This condition is satisfied by either $R = 0$, or $dR = 0$. Hence the sphere will be in equilibrium at points where the resultant magnetizing force vanishes; where it is a maximum or minimum; or where it is stationary in value.

8. A position of stable equilibrium will be such that R^2 diminishes in every direction from it; and hence, if there be any point, external to the magnet, at which the resultant force has a maximum value, it would be a position of stable equilibrium for a small ball of soft iron, and any other position of equilibrium is essentially unstable.

9. According to Mr. Faraday's recent researches, it appears that there are a great many substances susceptible of magnetic induction, of such a kind that for them the value of the coefficient i is negative. These he calls diamagnetic substances, and, in describing the remarkable results to which his experiments conducted him with reference to induction in diamagnetic matter, he says: "all the phenomena resolve themselves into this, that a portion of such matter, when

under magnetic action, tends to move from stronger to weaker places or points of force."* This is entirely in accordance with the result obtained above; and it appears that the law of all the phenomena of induction discovered by Faraday with reference to diamagnetics may be expressed in the same terms as in the case of ordinary magnetic induction, by merely supposing the coefficient i to have a negative value.†

10. In the case of a diamagnetic sphere, the consideration of the stability or instability of equilibrium in different positions, is extremely interesting. Thus, at a point where R^2 is a minimum, a small sphere of diamagnetic matter will be in stable equilibrium; and this is actually the case at any point for which the force vanishes: even if we take into account the weight of the sphere, it is readily shewn that stable positions of equilibrium may exist. Thus a hollow cylindrical bar-magnet (if sufficiently powerful), held with its axis vertical, would support a small diamagnetic sphere in a position of stable equilibrium at a point in the axis, a little below the lower end of the magnet. For, considering different points in the axis, we perceive that there is one below the lower end (at a distance $= \frac{a}{\sqrt{2}}$, if a, the radius of the cylinder, be very great compared with its thickness, and very small compared with its length, and if the distribution of magnetism be uniform) at which the resultant force is a maximum. If, on moving a small diamagnetic sphere upwards from this position, we arrive at a point where the force urging it upwards is greater than the weight, and then let it move freely from rest, it will oscillate about a position of stable equilibrium. It will probably be impossible ever to observe this phenomenon, on account of the difficulty of getting a magnet strong enough, and a diamagnetic substance sufficiently light, as the forces manifested in all cases of diamagnetic induction hitherto examined are excessively feeble.

11. A very curious phenomenon might readily be observed, according to the results given above, by placing two bar-magnets, with similar poles in the neighbourhood of

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* *Experimental Researches*, § 2418.

† The law of induction in a mass of any form, whether of magnetic or diamagnetic matter, may be stated as follows. Let R be the magnetic force upon a point within an infinitely small spherical surface, described round a point P in the mass, resulting from the magnetism of all the matter external to this surface. The intensity of the magnetism at P is equal to $\frac{3}{4}\pi i R$, and the direction is that of the resultant force R.

a ball of soft iron allowed to move in a horizontal straight line (suspended in such a manner that any motion which can take place is in a circle of considerable radius). Thus if a pole, S, of a bar-magnet which we may regard for simplicity as very long and thin, be held in the neighbourhood, the ball will be drawn towards the point A, in which a perpendicular from S meets the line of motion, and A will therefore be a position of stable equilibrium. If now a pole S', of an equally powerful magnet, be presented and held at an equal distance in SA produced, A will become an unstable position; and if the ball be placed in its line of motion, at any distance from A less than $\frac{SA}{\sqrt{2}}$, it will be *repelled* from A, although either magnet alone would cause it to move towards this point.

12. The result obtained above affords the true explanation of the phenomenon observed by Faraday, that a thin bar or needle of a diamagnetic substance, when suspended between the poles of a magnet, assumes a position across the line joining them. For there is no tendency for such a needle to arrange itself across the lines of magnetic force; but, as will be shewn in a future paper, if the needle be very small compared with the dimensions and distance of the magnet (as is the case, for instance, with a bar of any ordinary dimensions, subject only to the earth's influence), the direction it will assume, when allowed to turn freely about its centre of gravity, will be that of the lines of force, whether the material of which it consists be diamagnetic, or magnetic matter such as soft iron. Thus Faraday's result is due to the rapid decrease of magnetic intensity round the poles of the magnet, and to the length of the needle, which is considerable compared with the distance between the poles of the magnet; and is thus explained by the discoverer himself. (§ 2269) "The cause of the pointing of the bar, or any oblong arrangement of the heavy glass, is now evident. It is merely a result of the tendency of the particles to move outwards, or into the positions of weakest magnetic action.* The joint exertion of the action of all the particles brings the mass into the position which, by experiment, is found to belong to it."

St. Peter's College, May 13, 1847.

* The extreme feebleness of the diamagnetic action, on account of which any small sphere or cube of the matter will experience very nearly the same force as if all the rest were removed, seems fully to justify this explanation.

MATHEMATICAL NOTES.

I. *Note relative to Mr. Newman's paper on Logarithmic Integrals of the Second Order.*

After these pages had passed through the press, Mr. Cayley mentioned to the Editor that nearly the same subject had been treated in *Crelle's Journal* (Vol. xxx. 1840), by Professor Kummer. After a rapid perusal, I can only add that this is certainly true, and that *many* of the properties above investigated have been discovered, and some others besides. Whether some of mine are not wholly new, I am unable to assert positively, by reason of the great difference of notation; nevertheless I believe that several of my equations concerning Λ and ʖ are not contained in Professor Kummer's investigations.

He states, that the integral $\int F_1 x . \log F_2 x . dx$ was treated by *Hill* in the *Journal der Mathematik*, Band III., and the integral $\int \log (1 + 2x \cos \alpha + x^2)\, x^{-1}\, dx$, in a separate Latin treatise, by the same, in 1830. Kummer has enlarged on *Hill*, whose labours he regrets are so little known. It is curious that neither Kummer nor Hill seem to have known of *Spence's* integral, while virtually treating of the same under the form $\int (1 + x)^{-1} \log (\pm x) . dx$. It appears moreover from Kummer (p. 220), that *Clausen* has actually tabulated my integral ʖ in p. 298 of *Crelle's Journal*, Vol. VIII., under the form $-\int_0 \log (\pm\, 2 \sin \tfrac{1}{2} a)\, da$.

Professor Kummer conceives of the general integral under the form $\int F_1 x \int F_2 x dx . dx$; and he has also extended his views to the third, fourth, fifth, &c. orders of rational integrals (for this appears to be the more appropriate title), and has exhibited in them integrals which are analogous to those of the second order.

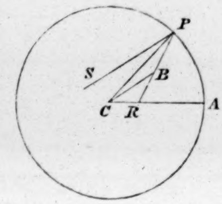
F. W. NEWMAN.

May 8*th*, 1847.

II. *On the Caustic by Reflection at a Circle.*

[*To the Editor.*]

A paper by Mr. Cayley, under the above title, having been published in the last number of your *Journal*, it appears to me that both M. de St. Laurent and Mr. Cayley have overlooked the admirably symmetrical solution of the problem given by Lagrange in the *Mem. de Turin*. Thinking that some of your correspondents may be interested in it, I beg to send you a translation.



Let B be the luminous point, RBP an incident, and PS a reflected ray; CA a fixed radius $ACP = \alpha$, $ACB = \epsilon$, reciprocal of $CB = c$, reciprocal of $CP = a$. The equations of the incident and reflected ray where $u = \frac{1}{r}$ may be written

$$u = A \sin \theta + B \cos \theta, \text{ incident ray,}$$

$$u = A \sin (2\alpha - \theta) + B \cos (2\alpha - \theta), \text{ reflected,}$$

the conditions for determining A and B being

$$a = A \sin \alpha + B \cos \alpha,$$

$$c = A \sin \epsilon + B \cos \epsilon;$$

whence $A = \dfrac{a \cos \epsilon - c \cos \alpha}{\sin (\alpha - \epsilon)}, \quad B = \dfrac{c \sin \alpha - a \sin \epsilon}{\sin (\alpha - \epsilon)}$.

Substituting these values, the equation of the reflected ray becomes

$$a \sin (2\alpha - \theta - \epsilon) = u \sin (\alpha - \epsilon) + c \sin (\alpha - \theta);$$

from which and its differential with respect to the arbitrary parameter α, the equation to the caustic or envelope of the reflected rays will be found by eliminating α.

In this, α being the only quantity treated as variable in the differentiation, let $2\alpha - \theta - \epsilon = 2\phi$,

therefore $$\alpha = \phi + \tfrac{1}{2}(\theta + \epsilon),$$

and the equation becomes

$$a \sin 2\phi = u \sin \{\phi + \tfrac{1}{2}(\theta - \epsilon)\} + c \sin \{\phi - \tfrac{1}{2}(\theta - \epsilon)\}.$$

Make $$P = \frac{(u + c) \cos \frac{1}{2}(\theta - \epsilon)}{2a},$$

$$Q = \frac{(u - c) \sin \frac{1}{2}(\theta - \epsilon)}{2a}.$$

Also $$x = \frac{1}{\cos \phi}, \quad y = \frac{1}{\sin \phi},$$

and the equation becomes

$$Px + Qy = 1,$$

with the condition $$x^{-2} + y^{-2} = 1.$$

Hence $$P = \lambda x^{-3},$$

$$Q = \lambda y^{-3}.$$

Multiplying by x and y, and adding, we find $\lambda = 1$;

therefore $$x^{-2} = P^{\frac{2}{3}}, \quad y^{-2} = Q^{\frac{2}{3}}.$$

Hence $$P^{\frac{2}{3}} + Q^{\frac{2}{3}} = 1;$$
or restoring the values of P and Q,

$$\left\{(u+c)\cos\tfrac{1}{2}(\theta-\epsilon)\right\}^{\frac{2}{3}} + \left\{(u-c)\sin\tfrac{1}{2}(\theta-\epsilon)\right\}^{\frac{2}{3}} = (2a)^{\frac{2}{3}},$$

the equation of the caustic.

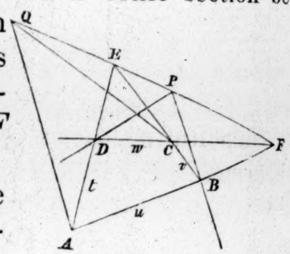
This equation, rationalized and transformed to rectangular coordinates, is identical with that of M. de St. Laurent.

PETER SMITH.

British Museum, April 24, 1847.

III. *Solution of a Problem from the Senate-House Papers for* 1847.

LET $ABCD$ be a quadrilateral, and let a conic section be described about it and tangents drawn at A, B, C, D. Let the opposite sides intersect in E, F, and the opposite tangents in P, Q. To prove that P, E, Q, F are in the same straight line.



Let $u=0$, $v=0$, $w=0$, $t=0$ be the equations to AB, BC, CD, DA, respectively.

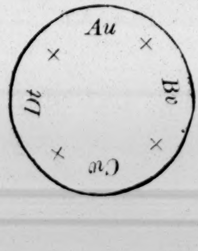
The equation to the conic described about $ABCD$ may be expressed by
$$uw = vt \ldots\ldots\ldots\ldots\ldots (1).$$

For this will be an equation of the second order, since u, v, w, t are linear in x and y. This equation is also satisfied by $u=0$, $t=0$; $u=0$, $v=0$; $v=0$, $w=0$; $w=0$, $t=0$; and therefore the curve passes through $ABCD$. Moreover the general form $ax+by+c$ being taken for each of the functions u, v, w, t, we may conceive each to have been multiplied by an arbitrary constant previously to combination in (1), and therefore the equation (1) will have all the generality possible.

Now let A, B, C, D be four constants such that
$$Au + Bv + Cw + Dt = 0 \text{ identically} \ldots\ldots (2).$$

Then the equation $Au + Cw = 0$ is the same as $Bv + Dt = 0$. But $Au + Cw = 0$ represents a line through F, and $Bv + Dt = 0$ one through E, and hence either equation represents EF.

Arrange the terms of (2) in circular order, thus



Then on taking the combinations u, v; v, w; w, t; t, u, combine each small letter with the adjacent capital, thus

$$\begin{aligned} Du + Cv &= 0, \quad * \\ Av + Dw &= 0, \\ Bw + At &= 0, \\ Ct + Bu &= 0. \end{aligned}$$

These are the equations to the tangents at B, C, D, A, respectively.

Hence for the point P we have simultaneously

$$Du + Cv = 0, \quad Bw + At = 0.$$

Multiply the first by AB and second by CD, and add

$$BD(Au + Cw) + AC(Bv + Dt) = 0,$$

which is satisfied by either of the identical equations $Au + Cw = 0$, $Bv + Dt = 0$ to EF, and therefore the point P is in EF. Combining the other two, we have the point Q in EF, therefore P, E, Q, F all in a straight line.

Cor. u, v, w, t being proportional to the perpendiculars from any point in the curve on the respective sides of $ABCD$, it appears from the equation $uw = tv$ that the product of perpendiculars on two opposite sides is proportional to the product of perpendiculars on the two other sides.

February 17. G. W. H.

Note. Let O be the intersection of AC, BD. In general if O be any point, and through it there be drawn lines meeting the conic in A, C and B, D if AD, BC meet in E, and AB, CD in F, EF is the polar of O, which is the case therefore in the figure. But P, Q are the poles of BD, AC respectively; hence these points lie in EF, or P, Q, E, F are in the same straight line.

March 27. A. C.

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* That $Du + Cv = 0$ is a tangent may be shewn thus:

$$uw = vt = -\frac{v}{D}(Au + Bv + Cw),$$

or $\quad Duw + v(Au + Bv + Cw) = 0.$

Let $u = \mu v$ be the tangent at B,

therefore $\quad D\mu vw + v(Au + Bv + Cw) = 0,$

or $\quad v = 0$ and $D\mu w + Au + Bv + Cw = 0.$

But if $u = \mu v$ be a tangent, it must satisfy the equation *twice*. Hence

$$D\mu + C = 0, \quad \mu = -\frac{C}{D},$$

and then $\quad (A\mu + B)v = 0$, or $v = 0$ a second time.

Hence $u = -\frac{C}{D}v$, or $Du + Cv = 0$ is the equation to the tangent at B. Similarly the other equations represent tangents.

IV. *On a System of Magnetic Curves.*

Let λ be the potential produced by a magnet symmetrical about an axis OX at a point $P(x, y)$. The magnetic curves, or lines of force, being the orthogonal trajectories of the surfaces for which the potential is constant, will lie in planes through OX; and the system in the plane YOX will be the orthogonal trajectory of the system of curves, λ = const. Their equation, as was shewn in a paper "On the Equations of Motion of Heat referred to Curvilinear Coordinates" (vol. IV. p. 40), is

$$\int y\left(\frac{d\lambda}{dy}\,dx - \frac{d\lambda}{dx}\,dy\right) = C \text{ .... } (1).$$

As an example, let λ be due to two small needles placed in the line OX, at points M, M'; so that we may take

$$\lambda = \frac{\mu(x-f)}{\{(x-f)^2+y^2\}^{\frac{3}{2}}} + \frac{\mu'(x-f')}{\{(x-f')^2+y^2\}^{\frac{3}{2}}} = \frac{\mu(x-f)}{\Delta^3} + \frac{\mu'(x-f')}{\Delta'^3}.$$

By integration we find, from (1),

$$\frac{\mu y^2}{\Delta^3} + \frac{\mu' y^2}{\Delta'^3} = C,$$

for the equation of the system of magnetic curves.

If we take as a particular case, $C = 0$, we find $y^2 = 0$, which shows that the axis is a line of force; we have also, for another branch, corresponding to the same value of C,

$$\frac{\mu}{\Delta^3} + \frac{\mu'}{\Delta'^3} = 0.$$

As Δ and Δ' are essentially positive in the physical problem, this can only be satisfied if μ and μ' have different signs. For instance, if $\mu = 1$, $\mu' = -m$, we have

$$\Delta' = m^{\frac{1}{3}}\Delta.$$

The locus of this equation is, as is well known, a circle, which may be described thus. Divide MM' in A, and produce it to A_1, so that

$$M'A = m^{\frac{1}{3}}.MA \text{ and } M'A_1 = m^{\frac{1}{3}}.MA_1;$$

on AA_1 as diameter describe a circle.

This result was suggested to me by the solution of a corresponding problem (of much greater interest however) in fluid motion, verbally communicated to me by Mr. Stokes.

WILLIAM THOMSON.

St. Peter's College, May 18, 1847.

ON PRINCIPAL AXES OF A BODY, THEIR MOMENTS OF INERTIA AND DISTRIBUTION IN SPACE.

By Richard Townsend.

(*Continued from* p. 171.)

These properties not only establish to a certain extent the analogy which we conceive to exist between the relation which connects these two unique developable surfaces with the two component developable systems into which every system of principal axes subject to a single condition may be always resolved, and the relation which connects the unique particular solution of an ordinary differential equation with the whole system of particular integrals of the same, but also they enable us to form a tolerably clear conception of the nature and position of the two conjugate systems of curves, the lines of contact and the lines of regression of the two developable systems on each sheet of the envelope, in all the different cases which may variously present themselves; viz. according as the two different curves which separate the available from the untouched regions of either sheet are both real and single or both real and consisting of two or more detached curves, or when one is imaginary and the other real and single or real and consisting of detached curves, or, finally, when they are both altogether imaginary. And here we may observe that it will be only necessary to consider in any particular instance what takes place on one sheet alone, for the corresponding curves of separation on the two sheets are obviously both at the same time real or at the same time imaginary, and if real are both of the same nature with respect to the mere number of separate curves they consist of, and that is all with which we need ever be concerned in the question before us.

First, then, let the separating curves of both species on one of the sheets of the envelope be both real and single; then will they always one or both either return into themselves, or else go off in both directions and meet at infinity: and moreover, since from the above properties both systems of lines, of contact and of regression, are such that every individual line of each has one or more points on both the separating curves, the available zone of the sheet bounded by these two curves must, in order to have that possible, be a continuous portion of the surface; hence, if the sheet be closed, that zone must necessarily be the closed belt occupying the interval between the two curves and not the two separated caps,

but if, on the contrary, it be open and extend to infinity in both directions, the infinite belt consisting of the two zones which, separated by the belt of finite breadth continuous or discontinuous lying between the two curves, extend each in opposite directions and meet at infinity, is not for the same reason excluded from being sometimes the available portion of the sheet; this is evident, for continuity is never broken or even virtually interrupted by mere passage through infinity.

Hence when the two bounding curves on the two sheets of the envelope are both real and single, the systems of lines of both species on each sheet, of contact and of regression, will cover the whole continuous zone included between these two curves: every line of both systems will have always one, often two or more, and sometimes even an infinite number, but in all cases not only for one but for both systems, the same number of cusps of the ramphoid species all placed on one of the two bounding curves, and also will have always the same number, whatever that may be, of points of contact with the other bounding curve, the points of contact placed alternately each on the interval between two successive cusps. These two systems of lines therefore, in all such cases, either will consist both of a number of closed or of open curves having each a cusp of the ramphoid species pointing outwards from the curve, such in the closed case as would be the common cardioid if its cusp were reversed, and in the open case of the same nature as the evolute of the common parabola; or they will consist both of two, of more, or of an infinite nnmber of arches placed side by side with each other and forming a curve of the same nature with the common cycloid: but in all cases the two bounding curves will be each the envelope of the whole system of lines of one species and the locus of the whole system of cusps of the other, the points of contact one or more of each line of either system coinciding always with the cusps one or more of the corresponding line of the other system.

Such being the nature and the distribution of the two systems of lines in the case when the two different separating curves on the two sheets are both real and single, the case when they are both real and consist of two or more detached curves presents no additional difficulty; for then, in case of one belt on which the lines are distributed as above, there will exist on each sheet two or more continuous belts of the same nature upon which the two conjugate systems of lines, both of the same nature as in the former case but divided

each into as many detached systems as there are separate belts, will be distributed in a manner exactly similar.

Such is the case when both bounding curves consist of the same number of detached curves: when however the number is not the same for both, then will only some of the detached systems of lines fall under the head already considered, and the others must be referred to some one or other of the following cases.

In the case when one of the two curves on each sheet is imaginary and the other real and single, then obviously the modification from the above general type will be, that one of the two systems of lines will have no cusps and the other will have no envelope, but in other respects their nature and their distribution will be exactly similar, the point or points of contact of each line of the enveloped system, whatever that may happen to be, with the single real curve envelope of that system coinciding always with the cusp or cusps of the corresponding line of the other system, and the two corresponding tangents at every coincident point being always conjugate to each other with respect to the surface; which indeed also is always the case all over the available portions of the envelope where a line of either system has a point in common with or, which is the same thing, intersects a line of the conjugate system.

But, as is much more frequently the case, one of the bounding curves being imaginary, when the one real curve on each sheet consists of two detached curves or of two or more distinct pairs of detached curves, then, for the reasons before explained, will the available intervals between each pair be all continuous and generally, of finite breadth, though returning back into themselves or going off to infinity indifferently as the case may be; the two systems of lines will be divided both into as many detached portions as there are belts of this nature, that is, as there are pairs of curves, and of the portions on each belt every line of one system will touch the two bounding curves of that belt each at least in one point, but often in more, and sometimes in an infinite number, but always both in the same number of points; and every line of the other system will have as many pairs of cusps as the lines of the former have pairs of points of contact, every pair of cuspal points of every line of that system lying on the double curve and coinciding with the corresponding pairs of points of contact of the corresponding line of the former system, and the two corresponding tangents at every individual point of coincidence, being always

conjugate to each other with respect to the surface. If the belt, whether closed or going off to infinity, be, as is most frequently the case, of finite breadth, then on every line of both systems a point of inflexion must necessarily exist, for the latter system, between every pair of successive cusps, and for the former system, between every pair of successive points of contact; and whenever the number of points of contact of each line of that system is large or infinite, the lines themselves will obviously be all of an undulating nature, consisting like the sinusoid of a number of alternate elevations and depressions, with an equal number of points of inflexion ranged alternately between, and forming the points of transition from the depressed to the elevated arches, while in the same case the lines of the other system will be of a nature altogether different, consisting certainly each of a number of successive portions all of the same kind, but in place of an undulating appearance presenting two rows of sharp cusps, pointing alternately in opposite directions, and placed between these containing also a row of points of inflexion ranged alternately between every successive pair of opposite cusps, the cusps in each row being equal in number and pointing all outwards, and the number of points of inflexion being equal to the total number of cusps, and therefore to double the number contained in each row.

Finally, in the case when the two separating curves are both altogether imaginary, then will the two sheets of the envelope be both altogether available, and the two systems of lines on each, the lines of contact and the lines of regression, will both be altogether changed in character, neither system will admit of an envelope, and no line of either system will be endowed with a cusp, but both systems of lines will completely cover the whole surface, and will either or both on each sheet, as the case may be, either return back into themselves or extend in both directions and meet at infinity; and at every point taken arbitrarily on either sheet there will cross two distinct lines of each system intersecting at an angle of finite magnitude, the two tangents to the two intersecting lines of either system being respectively the two conjugates with respect to the surface of the two tangents to the two corresponding intersecting lines of the other system.

This last property is generally true, whatever be the nature of the envelope and of the two bounding curves, all over the available regions of that surface; for at all the points of these regions there will obviously cross as many lines of regression as there are tangents which are principal axes, such tangents

giving the directions in which the lines of that system, whatever be their number, diverge from each point: but the cone of principal axes diverging from every point being always of the second order, there will generally diverge from each point two, and there can diverge but two tangents which will be principal axes, and therefore two and but two lines of regression, and consequently also two and but two lines of the conjugate system. Hence, to state the property before us in its greatest generality, we may say, that through every point on each sheet of every surface envelope of a system of principal axes subject to a single condition, including not only the available but also the untouched region or regions of that surface and the separating curve or curves, there pass always two lines of each species, of regression and of contact, the pairs being simultaneously both real, both imaginary, or both coincident, as the case may be.

It not unfrequently happens, especially on envelopes of which the two sheets are altogether available, that points exist for which the two tangent principal axes are conjugate to each other with respect to the surface: at all such points the two diverging lines of each species coincide obviously in direction with each other, and therefore if any envelope be such that the same takes place at every point, then for that surface will the two systems of lines of contact and of regression absolutely coincide with each other, that is, the same lines which form the curves of contact of one of the two component systems of developable surfaces, into which the enveloped system of axes may be resolved, will also form the curves of regression of the other component developable system; and conversely, if either sheet of an envelope possess the property that its two conjugate systems of lines coincide with each other, then at every point of that sheet will the two tangent principal axes be always conjugate to each other with respect to the surface. We shall see as we proceed, that there exists in every body a very extensive class of systems of envelopes, the class containing an infinite number of systems, and each system containing an infinite number of envelopes, all possessing these unusual properties, that both their sheets are entirely available, and that their two systems of lines on each sheet, of contact and of regression, coincide with each other all over the whole extent of the surface. Moreover, if on a surface of this nature, besides crossing at every point of the surface in directions which are always conjugate to each other, the two intersecting sets of lines of the same species intersect everywhere two and two at right

angles, then for that surface will these two sets of lines be the opposite systems of lines of curvature: the classes of surfaces to which we have just now alluded possess also that additional property.

We may now, by means of the above properties, find readily upon any surface whatever, given or arbitrarily assumed in a body, the two different bounding curves which separate from each other its regions of real and imaginary contact with the principal axes of the body; and moreover we can then know immediately the nature and distribution of the whole system of curves upon the surface along every one of which the developable circumscribing the surface will have all its edges principal axes, and also the nature and distribution of that system of lines traced out on the same which will all possess the property that their systems of tangents will be all principal axes. For we have but to describe the second sheet of the surface envelope of the system of principal axes of the body which are all tangents to the given surface, that sheet will intersect the surface in one of the curves required; to find the second we have but to describe the developable surface circumscribing the two sheets, that developable will touch the given surface along the other curve required; and then, finally, to find the two conjugate systems of lines we shall merely have to apply the preceding general principles, which, whenever we have the two bounding curves, put us at once in possession of the nature, position, and distribution of the two required systems of lines.

Hence we see that on every surface assumed arbitrarily in a body there exist two remarkable curves loci of two distinct systems of points, for both of which the two tangent principal axes will at every point coincide with each other; that one of these curves is always the envelope of the corresponding system of coincident principal axes, and that the other is always the envelope of the system of tangents conjugate to its corresponding system of coincident principal axes; that through every point of the surface there pass two lines of regression and two lines of contact, all real and different, all imaginary, or two and two coincident, as the case may be; and that these two systems of lines, whose nature and distribution different for different surfaces is in all cases determinable from the nature and positions of the two bounding curves, are on every surface whatever always conjugate to each other.

On every surface upon which the two bounding curves are both real—since at all points on one of them the two

tangent principal axes intersect at an evanescent angle, while at all points on the other they intersect at an angle equal to two right angles—there must always exist on the intervening region of real contact between the two curves a line of points for all of which the two tangent principal axes intersect at right angles: this line, which is by no means confined to that particular class of surfaces, but is also found frequently upon surfaces of the other two classes, and which is in all cases a very remarkable curve, we shall often again have occasion to notice; and in the sequel we shall see moreover that in every body there exists an infinite number of classes of surfaces, for which every surface of each class possesses at every point that peculiar property.

Every system of principal axes subject to a single restricting condition being resolvable, as appeared from the preceding principles, into either of two and of but two component systems of developable surfaces, it is obvious that through any individual axis of the system there can pass one and but one developable of each component system; hence we know respecting every such system of principal axes, that of the infinite number of axes of the system which pass all infinitely near to any one individual axis of the same, four and but four will in general intersect that axis, which four will consist of two distinct pairs of axes, the axes of each pair lying at opposite sides of the original axis and ultimately intersecting it at the same point, so that though there are always four intersecting axes, there are never but two points of intersection. This property is much more general, and holds not only for every such system of principal axes, but equally for every system of right lines in space which are subject to two independent restricting conditions whatever be their nature; for if we retrace the steps by which the preceding principles were established, we shall see that every such system of right lines possesses always an envelope of the same nature as that we have been endeavouring to describe, and that when the envelope is real the system may always be resolved into either of two and of but two different and distinct systems of developable surfaces. Hence deducing from these properties the above general inference for every system of right lines whatever which are subject to two independent restricting conditions, we arrive in substance at a celebrated theorem of the illustrious Monge, much admired by the no less distinguished Professor Chasles, who in the Appendix to his History of Geometry has furnished us with the polar reciprocal correlative property.

Indeed, whatever has been said in the present article respecting a system of principal axes subject to a single restricting condition, holds more generally and with scarcely an exception for every system of right lines in space subject to two independent restricting conditions, the only difference between the particular and the more general case being, that in the former one of the two restricting conditions is given and is always the same, while in the general case they are both variable and arbitrary. It was for this reason that we have dwelt so long on this (which is far from being the most interesting) part of our subject, because that we have all along been implicitly discussing the more general question respecting the management and properties of a system of right lines subject to two conditions, the nature and properties of the different systems of rule surfaces into which such a system of lines may be resolved, the nature and varieties of the surface, their envelope, and the consequent nature, position, distribution, and varieties of the two conjugate systems of generating curves on each sheet of that surface, the lines, namely of contact and of regression of the two component systems of developable surfaces into which every such system of right lines in space may be always resolved.

One property indeed (and it seems to be the only one) requires a different method of establishment, viz. that at every point of the envelope two and but two right lines of the system enveloped can touch that surface, and therefore that through every point on the same there can pass two and but two lines of each system of contact and of regression; for it is not every pair of conditions for which the cone resulting from one or either of them will be always of the second order, and besides that the property itself is not without exception true; that it holds however in the general case also, and holds moreover for the most part though not universally, may be easily shewn as follows: Let a system of right lines subjected to one of the two conditions, whatever they be, be constrained to pass all through a point, they will generate a cone of some order or other, let then another system of right lines subjected to the other condition be constrained to pass all through the same point, they will generate another cone: the intersecting sides of these two cones will evidently be the only lines passing through the point which fulfil at once the two conditions, and it is obvious that in general no three of them, except accidentally or in particular cases, will ever lie in the same plane. Hence we see that even one

right line of a system subject to two independent conditions cannot always be drawn through a given point so as to lie in a plane given or drawn arbitrarily through that point, and that out of the whole system of planes which contain some one of the limited number of lines diverging from that point which fulfil the two conditions, not one will in general ever contain more than two. Again, let a system of right lines subjected to one of the two conditions be constrained to lie all in a plane, they will envelope a curve in that plane of some order or other; let then another system subjected to the other condition be constrained to lie all in the same plane, they will envelope another curve: the common tangents to these two curves will evidently be the only lines lying in that plane which fulfil at once the two conditions, and, as in the other case, it is obvious that in general no three of them, except accidentally or in particular cases, will ever pass through the same point. Hence we see that even one right line of a system subject to two independent conditions cannot always be drawn in a given plane so as to pass through a point given or arbitrarily assumed in that plane, and that out of the whole system of points which lie on some one of the limited number of lines lying in that plane which fulfil the two conditions, through no one will there in general ever pass more than two.

From these two general properties combined we see immediately that between a point and a plane passing through it, or between a plane and a point lying in it, a very close and intimate connexion must exist with respect to a system of right lines subject to two independent conditions, in order that two lines of the system should both at the same time pass through the point and lie in the plane, and also that in general more than two lines of the system could never, except accidentally, at the same time pass all through a point and lie all in a plane. It is obvious that the necessary connexion must exist at every point all over the available regions of the surface which envelopes that system of right lines, between every individual point and the corresponding tangent plane to the surface, and that therefore at every point of the envelope, whatever be its nature, there will generally touch that surface two and but two right lines of the system enveloped, both real or both imaginary as the case may be.

But here we must be cautious, for this result though very generally is by no means universally true: this is obvious from the following considerations, which also explain at the same time the general cause of failure in particular cases.

Suppose that one of the two curves in any plane whose common tangents were the only lines in that plane, fulfilling at once the two restricting conditions had a double point, nodal or conjugate, or more generally a multiple point of any order whatever, then would all the tangents drawn from the multiple point to the other curve come under the head of common tangents to the two curves, and therefore in such cases exceptions would exist to the general rule that no more than two right lines of the system fulfilling the two conditions could at the same time pass through the same point and lie in the same plane. A very general and extensive class of exceptions of this nature is to be found in a system of right lines subject to the two independent conditions of touching two given surfaces; for if we draw any plane whatever intersecting both surfaces in a pair of curves, then will the common tangents to these curves be the lines in that plane which fulfil the two conditions; but in the particular case when the common intersecting plane touches either of the surfaces, then will the point of contact be a double point of the curve in which it intersects that surface, and obviously all tangents from that point to the other curve will fulfil the two conditions. The same manifestly may be said also of every system of right lines subject to the two conditions of having double contact with a single given surface, such being in fact but particular cases of the former, the two given surfaces being conceived as coming together and coinciding in one. Hence in the extensive class of cases where the complete envelope itself is given and where the restricting conditions are in contact with both its sheets, or double contact with its single sheet if it consist of only one, the above general property fails, and more than two tangents at the different points of the surface could in general be drawn fulfilling the two conditions.

In every case when we have a system of principal axes, such as we have been considering, subject to a single condition, the corresponding system of principal planes will, obviously, also envelope a surface; and to find that envelope when the restricting condition is given, we may proceed exactly on the principles already described—for the introduction of an arbitrary condition resolves the system of axes into a multitude of groups each forming a surface gauche or developable as the case may be, and therefore divides the whole corresponding system of planes into an infinite number of smaller systems having each for its envelope a developable surface, and of this system of developables the envelope is of

course that of the whole original system of planes: in order therefore to find its equation we have but to investigate, after the manner indicated in (31), the equation of the developable corresponding to one of the smaller systems of axes, determined by an arbitrarily introduced condition, and from that equation containing the parameter of that system and therefore expressing the whole system of developables, proceed then in the usual way to find the envelope of that system of surfaces, that is, of the whole system of principal planes.

An obvious but very particular example illustrative of the principles discussed in this article is afforded by every system of principal axes which are normals all to the same surface of the second order confocal with the ellipsoid of gyration: here they are restricted by but a single condition; here they may be resolved into a multitude of smaller systems forming each a surface, and this resolution may be performed in an infinite number of different ways, for by simply introducing at random the equation of a surface containing a parameter, we shall then, by varying that parameter, have the original surface of the second order covered with a multitude of curves, and we may take for our resolved systems those groups of axes which pass each through one of these curves: by introducing the equations of the confocal system itself the system of curves will be the lines of curvature on the original surface, the corresponding resolved systems of axes will all form developable surfaces, and there being two and but two different and distinct systems of lines of curvature, each covering the whole original surface, there are therefore two and but two different and distinct systems of developable surfaces into which this system of axes may be resolved. Again, all the different resolved systems will have one and the same envelope, consisting of two different and distinct sheets, each touched by every axis of the system enveloped and generally at different points of contact, the "surface of centres" of the original surface; and, finally, the whole corresponding system of principal planes will also envelope a surface, the original surface itself.

(To be continued.)

NOTES ON DESCRIPTIVE GEOMETRY. NO. II.

By T. S. Davies, F.R.S., F.S.A.

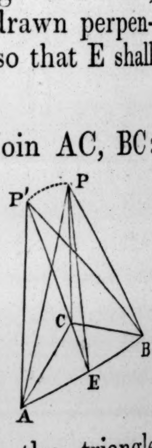
The relative magnitudes, as to greater and less, of an angle and its orthographic projection, have been *assumed* by most if not all writers on the subject of these notes, without investigation. That assumption is, that "the projection is *always* greater than the projected angle." In some cases this is true, in others not: and the object of this note is to examine all the cases that can arise.

The course here employed will have the further advantage of furnishing a more concise and intelligible demonstration of *Euc.* XI. 21, than that which has descended to us from the Greeks, and which is generally employed by modern writers. On this account, the investigation is written in the ordinary language of geometry rather than in the special language of projection.

Problem. *Let* CD *be a line perpendicular to a plane* ABC *meeting the plane in* C; *and let* A, B, *be two points in that plane: it is required to find the point in* CD *to which if lines be drawn from* A *and* B, *they shall contain the greatest angle.*

The problem will divide itself into two general cases, adapted to the circumstances that CE being drawn perpendicular to the line AB, it shall meet AB in E so that E shall be in AB or in AB produced, respectively.

Case 1. Let E lie between A and B. Join AC, BC: then ACB is the greatest angle formed by lines from A, B, to meet in CD.



For let P be any other point in CD, and join PE; make EP′ equal to EP; and join AP′, P′B.

Then, since PCE is a right angle, it is greater than CPE; and hence PE is greater than CE, and P′ lies more remote from E than C does. The point C is therefore within the triangle AP′B; and (*Euc.* I. 21) the angle ACB is greater than AP′B.

Again (*Euc.* XI. 11), PE is perpendicular to AB, or AEP is a right angle. Whence the triangles APE, AP′E have the sides AE, EP equal to the sides AE, EP′, and their included angles AEP, AEP′ also equal: and therefore the angle APE is equal to AP′E.

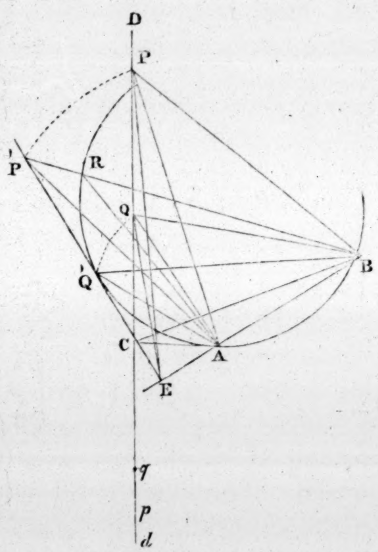
Similarly the angle BPE is equal to BP′E; and consequently the whole angle APB is equal to AP′B.

But ACB is greater than AP′B, and consequently greater than APB.

The same demonstration evidently applies when P is taken on the other side of the plane ACB: and when E falls at A or B, the demonstration becomes still more simple in its details.

Scholium. When PA, PB are the lines forming an angle to be projected on a plane which meets them in A and B, and when the perpendicular from P to the base AB does not fall in AB produced, the projection of the angle is greater than the angle itself. The general assumption referred to above is therefore always true in this case; and it will be seen that this is not the only case in which it is accurate.

Case 2. Let the perpendicular CE from C to AB meet AB *produced*, in E. Through A and B describe a circle to touch EC *produced* in the point Q′; in CD take Q so that EQ shall be equal to EQ′, and join AQ, QB. Then AQB is the greatest angle that can be formed by lines drawn from A and B to meet in CD, on the same side of the plane ACB.



For, let P be any other point in CD on the same side of the plane ACB; and in EC produced take EP′ equal to EP: the other lines being drawn as in the figure to join the several points already defined, and R being the intersection of P′B with the circle ABQ′.

Then since AQ′B, ARB are angles in the same segment of the circle AQ′B, they are equal; and since in the triangle P′AR the angle ARB is exterior, it is greater than AP′B. Wherefore AQ′B is greater than AP′B.

Again, by reasoning similar to that in the first case, it may be shewn that APB is equal to AP′B, and AQB equal to AQ′B; and hence it follows that AQB is greater than APB.

Scholium. In this case, then, the assumption is erroneous, as the orthographic projection represented by ACB is less than the projected angle AQB. This is the case when P′ is taken at C, and P coincides with P′.

In order to specify the varieties of this general case, one or two remarks may be usefully appended. Let us then suppose the plane ACB to be horizontal, and CD to be the part of the line *above* ACB; whilst the part *below* ACB is denoted by C*d*. Also, let C*q*, C*p* be equal to CQ, CP. Then,

(α). The angles made by inflecting lines from A and B to points in CD continually increase as the point to which the lines are inflected approach towards Q from D. After passing Q, the angles continually diminish till the point of inflection arrives at C. The angles then again increase till the point arrives at *q*; and, finally, in passing farther downwards the angles diminish incessantly. The extreme limits of the magnitudes of the angles in both directions is zero.

The problem, then, in this case has two maxima solutions and one minimum—using these terms in their modern mathematical sense.

(β). If a circle described through A, B, C, and cut the line EC produced in P′; and if P be constructed so that EP shall be equal to EP′: then the orthographic projection ACB of the angle APB will be equal to the angle APB itself.

(γ). If a circle be described through A, B to cut CE produced between C and Q′, in H′, and again in K′; and points H, K corresponding to them be taken in CD (these points are omitted in the figure to prevent confusion): then the angles AHB, AKB will be equal to one another. Whence two equal angles situated in different planes passing through AB may have the same orthographic projection. The same is obviously true of two angles A*k*B, A*h*B on the other side (or *below*) the plane ACB.

(δ). If the circle through A, B touching CE, touch it in C, the point Q coincides with C; and the angle ACB will be the greatest possible.

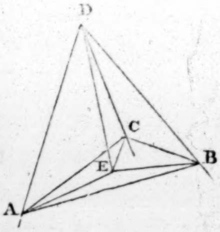
In this case, then, the ordinary assumption is correct as to the projection being greater than the projected angle. It is this which gives rise to the limitation expressed in the construction of the second case.

(ε). If the circle through A, B touch CE between C and *c* (*c* taken equidistant from E with C) the point determined by it is excluded from consideration; since no corresponding point could be constructed in CD, CE being the shortest line that can be drawn from E to CD. In this case, then, the usual assumption is also accurate.

EUCLID XI. 21. *Every solid angle is contained by plane angles, which are together less than four right angles.*

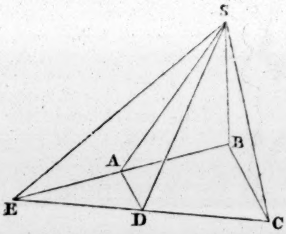
The demonstration of this theorem will only require the property established in the *first case* of the preceding proposition together with the preceding propositions of Euclid. It will be divided into two cases corresponding to Euclid's own division.

(a). Let the angle D be trihedral; take DA, DB, DC all equal; draw the perpendicular DE to the plane ABC; and join EA, EB, EC.



Then, since DEA, DEB, DEC are right angles, it readily follows from (I. 47) and the conditions of our construction, that AE, BE, CE are equal, and hence that E is the centre of the circle about ABC. Whence the perpendiculars from E to the sides of the triangle fall *between* the extremities of those lines, since those perpendiculars bisect the sides.

Wherefore ADB, BDC, CDA are respectively less than AEB, BEC, CEA; and hence their sum less than the four right angles to which AEB, BEC, CEA are equal.



(β). Let the angle S be tetrahedral, and produce two noncontiguous faces to meet in SE; also, let the system of planes be cut by any other plane, ABCDE.

Then, by the preceding case, the angles ESB, BSC, CSE are less than four right angles. But (XI. 20) the angle ASD is less than the two ESA, ESD; and hence the angles ASB, BSC, CSD, DSA are less than ESB, BSC, CSE; and therefore, *à fortiori*, less than four right angles.

(γ). In all other cases the sum of the plane angles may be shewn to become less and less as their number is increased; and hence, that the proposition is true in its most general form, as long as the dihedral angles are salient.

Royal Military Academy, Woolwich,
April 3, 1847.

ON THE THEORY OF ELLIPTIC FUNCTIONS.

By Arthur Cayley.

Adopting the notation of the *Fund. Nova.* except that for shortness $sn.u$, $cn.u$, $dn.u$ are written instead of sinam.u, cosam.u, $\Delta am.u$, let the functions $\Theta(u)$, $\mathrm{H}(u)$ be defined by the equations

$$\Theta u = \sqrt{\left(\frac{2KK'}{\pi}\right)}\, e^{\frac{1}{2}u^2.\left(1-\frac{E}{K}\right) - k^2.\int_0 du \int_0 du\, sn^2 u} \quad \ldots (1),$$

$$\mathrm{H}u = -ie^{-\frac{\pi(K'-2iu)}{4K}}\,\Theta(u+iK)\ldots\ldots\ldots(2),$$

it is required from these equations to express $sn.u$ in terms of the functions $\mathrm{H}(u)$, $\Theta(u)$. To accomplish this we have

$$\frac{d^2}{du^2}\log snu = \frac{1}{snu}\frac{d^2}{du^2}snu - \frac{1}{sn^2u}\left(\frac{d}{du}snu\right)^2$$

$$= -(1+k^2) + 2k^2 sn^2u - \left\{\frac{1}{sn^2u} - (1+k^2) + k^2 sn^2 u\right\}$$

$$= k^2 sn^2 u - \frac{1}{sn^2 u};$$

whence also

$$\frac{d^2}{du^2}\log snu = k^2 sn^2 u - k^2 sn^2.(u+iK').$$

If for a moment

$$\psi_{,}u = \int_0 du\, sn^2 u, \qquad \psi_{,,}u = \int_0 du \int_0 du\, sn^2 u,$$

$$\log snu = k^2\psi_{,,}u - k^2\psi_{,,}(u+iK') + Au + B.$$

Or writing $(-u)$ for u and subtracting, $\psi_{,,}u$ being an even function,

$$2Au = \pi i - k^2\psi_{,,}(iK'-u) + k^2\psi_{,,}(iK'+u),$$

or putting $u = K$,

$$2AK = \pi i - k^2\psi_{,,}(iK'-K) + k^2\psi_{,,}(iK'+K).$$

Now $\qquad sn^2(u+K) - sn^2(u-K) = 0,$

and therefore $\qquad \psi_{,}(u+K) - \psi_{,}(u-K) = 2\psi_{,}K,$

$$\psi_{,,}(u+K) - \psi_{,,}(u-K) = 2u\psi_{,}K;$$

or $\qquad \psi_{,,}(iK'+K) - \psi_{,,}(iK'-K) = 2iK'\psi_{,}K.$

Also $$E(u) = u - k^2\psi_{,}u,$$

or $$E = K - k^2\psi_{,}K, \quad \text{i.e. } \psi_{,}K = \frac{K}{k^2}\left(1 - \frac{E}{K}\right).$$

Hence $$A = iK'\left(1 - \frac{E}{K}\right) + \frac{\pi i}{2K},$$

$$\log snu = k^2\psi_{,,}u - k^2\psi_{,,}(u + iK') + uiK'\left(1 - \frac{E}{K}\right) + \frac{\pi ui}{2K} + B$$

$$= k^2\psi_{,,}u - k^2\psi_{,,}(u + iK') + \tfrac{1}{2}[(u + iK')^2 - u^2]\left(1 - \frac{E}{K}\right) + \frac{\pi ui}{2K} + B',$$

i.e. $$\log snu = \log\Theta(u + iK') - \log\Theta u + \frac{\pi ui}{2K} + B',$$

or, changing the constant,

$$snu = Ce^{\frac{\pi ui}{2K}}\frac{\Theta(u + iK')}{\Theta u}.$$

Now, to determine C, write $u - iK'$ for u; this gives

$$\frac{1}{ksnu} = Ce^{\frac{\pi i}{2K}(u - iK')}\frac{\Theta u}{\Theta(u - iK')}:$$

and again changing (u) into $(-u)$,

$$-snu = Ce^{-\frac{\pi ui}{2K}}\frac{\Theta(u - iK')}{\Theta u}:$$

whence, multiplying these last two equations,

$$C^2 = -\frac{1}{k}e^{-\frac{\pi K'}{2K}},$$

or $$C = \frac{1}{i\sqrt{k}}e^{-\frac{\pi K'}{4K}};$$

whence $$sn\,u = \frac{1}{i\sqrt{k}}e^{-\frac{\pi(K' - 2iu)}{4K}}\frac{\Theta(u + iK')}{\Theta u},$$

i.e. $$\sqrt{k}\,sn\,u = \frac{\mathrm{H}(u)}{\Theta(u)} \quad \ldots\ldots\ldots\ldots (3);$$

and the equations (1), (2) and (3) may be considered as comprehending the theory of the functions $H(u)$, $\Theta(u)$. The preceding process is, in fact, the converse of that made use of in the *Fund. Nova;* Jacobi having obtained for $sn.u$ an expression in the form of a fraction, takes the numerator of it for $H(u)$ and the denominator for $\Theta(u)$, and thence deduces the equations (1), (2), the intermediate steps of the demonstration being conducted by means of infinite series; the necessity of which is avoided by the preceding investigation.

I proceed to investigate certain results relating to these functions, and to the theory of elliptic functions which have been given by Jacobi in two papers, "Suite des notices sur les fonctions elliptiques," *Crelle*, tom. III. p. 306, and tom. IV. p. 185, but without demonstration.

In the first place, the equation

$$\frac{d^2\Sigma}{du^2} + 2u\left(k'^2 - \frac{E}{K}\right)\frac{d\Sigma}{du} + 2kk'^2\frac{d\Sigma}{dk} = 0 \quad \ldots . (4)$$

is satisfied by $\Sigma = \Theta(u)$ or $\Sigma = H(u)$. It will be sufficient to prove this for $\Sigma = \Theta(u)$, since a similar demonstration may easily be found for the other value. The following preliminary formulæ will be required:

$$k\frac{dK}{dk} = \frac{E}{k'^2} - K, \quad k\frac{dE}{dk} = E - K,$$

$$k\frac{dK'}{dk} = -\frac{E'}{k'^2} + \frac{k^2K}{k'^2}, \quad KK' - EK' - E'K = -\frac{\pi}{2},$$

which are all of them known.

Now, writing $\Theta(u)$ under the slightly more convenient form

$$\Theta u = \sqrt{\left(\frac{2Kk'}{\pi}\right)}\, e^{\int_0 du \int_0 du\, dn^2 u - \frac{1}{2}u^2 \frac{E}{K}},$$

we have $$\frac{d\Theta u}{du} = \left(\int_0 du\, dn^2 u - \frac{E}{K}u\right)\Theta u$$

$$= \left\{u\left(k'^2 - \frac{E}{K}\right) + k^2\int_0 du\, cn^2 u\right\}\Theta u,$$

$$\frac{d^2\Theta u}{du^2} = \left[dn^2 u - \frac{E}{K} + \left\{u\left(k'^2 - \frac{E}{K}\right) + k^2\int_0 du\, cn^2 u\right\}^2\right]\Theta u,$$

$$\frac{d\Theta u}{dk} = \left[\frac{1}{2Kk'}\frac{dKk'}{dk} - \tfrac{1}{2}u^2\frac{d}{dk}\frac{E}{K} + \int_0 du\int_0 du\frac{d}{dk}dn^2 u\right]\Theta u.$$

The success of the process depends upon a transformation of the double integral

$$\int_0 du \int_0 du \,\frac{d}{dk}\, dn^2 u.$$

To effect this we have

$$\frac{d}{dk}\, dn^2 u = -\,2k\, sn\, u \left(sn\, u + k\, \frac{d}{dk}\, sn\, u \right);$$

but, by a known formula,

$$k'^2 \frac{d}{dk}\, sn\, u = -\, k\, cn\, u\, dn\, u \int_0 cn^2 u\, du + k\, cn^2 u\, sn\, u\,;$$

whence $sn\, u + k\, \dfrac{d}{dk}\, sn\, u = \dfrac{1}{k'^2}\, sn\, u\, dn^2 u - k^2\, cn\, u\, dn\, u \int_0 du\, cn^2 u,$

or $\dfrac{d}{dk}\, dn^2 u = -\, \dfrac{2k}{k'^2}\,(sn^2 u\, dn^2 u - k^2\, sn\, u\, cn\, u\, dn\, u \int_0 du\, cn^2 u)$

$$= -\,\frac{2k}{k'^2}\left\{ sn^2 u\, dn^2 u + \tfrac{1}{2}k^2 \left(\frac{d}{du}\, cn^2 u\right) \int_0 du\, cn^2 u \right\};$$

whence $\displaystyle\int_0 du \int_0 du \,\frac{d}{dk}\, dn^2 u$

$$= -\frac{2k}{k'^2}\{\textstyle\int_0 du \int_0 du\, sn^2 u\, dn^2 u + \frac{1}{2}k^2 \int_0 du\, (cn^2 u \int du\, cn^2 u - \int du\, cn^4 u)\}$$

$$= -\frac{k}{k'^2}\{\textstyle\int_0 du \int_0 du\, (2sn^2 u\, dn^2 u - k^2\, cn^4 u) + \frac{1}{2}k^2 (\int_0 du\, cn^2 u)^2\}.$$

But $\dfrac{d^2}{du^2}\, sn^2 u = 2\,(cn^2 u\, dn^2 u - sn^2 u\, dn^2 u - k^2\, sn^2 u\, cn^2 u)$

$$= 2\,(k'^2 - 2sn^2 u\, dn^2 u + k^2\, cn^4 u)\,;$$

or, integrating,

$$sn^2 u = k'^2 u^2 - 2 \int_0 du \int_0 du\, (2sn^2 u\, dn^2 u - k^2\, cn^4 u)\,;$$

whence at length

$$\int_0 du \int_0 du\, \frac{d}{dk}\, dn^2 u = -\,\tfrac{1}{2}ku^2 + \tfrac{1}{2}\,\frac{k}{k'^2}\, sn^2 u + \frac{k^3}{2k'^2}\,(\textstyle\int_0 du\, cn^2 u)^2.$$

Also $\dfrac{d}{dk}\, Kk' = \dfrac{E - K}{Kk'}\,, \quad \dfrac{d}{dk}\,\dfrac{E}{K} = \dfrac{1}{kk'^2}\left\{ k'^2 \left(\dfrac{2E}{K} - 1\right) - \dfrac{E^2}{K^2}\right\},$

so that

$$\frac{d\Theta u}{dk} = \frac{1}{2kk'^2}\left\{ \frac{E}{K} - dn^2 u - u^2 \left(k'^2 - \frac{E}{K}\right)^2 - k^4\, (\textstyle\int du\, cn^2 u)^2 \right\} \Theta u.$$

And substituting these values of $\frac{d}{du}\Theta u$, $\frac{d^2}{du^2}\Theta u$ and $\frac{d}{dk}\Theta u$ in the equation (4) in the place of the corresponding differential coefficients of Σ, all the terms vanish, or the equation is satisfied by $\Sigma = \Theta(u)$, and similarly it would be satisfied by $\Sigma = \mathrm{H}(u)$.

Assume now $\omega = \frac{\pi K'}{K}$, $v = \frac{\pi u}{2K}$.

Then observing the equation

$$\frac{d}{dk}\frac{K'}{K} = \frac{1}{K^2 kk'^2}(KK' - KE' - K'E) = -\frac{\pi}{2K^2 kk'^2},$$

we have $\frac{d\Sigma}{du} = \frac{\pi}{2K}\frac{d\Sigma}{dv}$, $\frac{d^2\Sigma}{du^2} = \frac{\pi^2}{4K^2}\frac{d^2\Sigma}{dv^2}$,

$$\frac{d\Sigma}{dk} = \frac{v}{kk'^2}\left(k'^2 - \frac{E}{K}\right)\frac{d\Sigma}{dv} - \frac{\pi^2}{2K^2kk'^2}\frac{d\Sigma}{d\omega};$$

whence, substituting in the equation (4), this becomes

$$\frac{d^2\Sigma}{dv^2} - 4\frac{d\Sigma}{d\omega} = 0 \quad \ldots\ldots\ldots\ldots (5),$$

which is of course satisfied as before by $\Sigma = \Theta(u)$, or $\Sigma = \mathrm{H}(u)$, an equation demonstrated in a different manner (by means of expansions) by Jacobi in the Memoirs quoted.

Consider next the equation

$$\frac{d^2\Sigma}{du^2} - 2nu\left(k'^2 - \frac{E}{K}\right)\frac{d\Sigma}{du} + 2nkk'^2\frac{d\Sigma}{dk} = 0 \ldots (6),$$

(n being any positive integer number). Then, by assuming

$$\omega = n\frac{\pi K'}{K}, \quad v = \frac{n\pi u}{K},$$

we should be led as before to the equation (5). Hence, considering Θu or $\mathrm{H}u$ as functions of u and $\frac{K'}{K}$, the equation (6) is satisfied by assuming for Σ a corresponding function of nu and $\frac{nK'}{K}$. Let λ be the modulus corresponding to a transformation of the n^{th} order; then Λ, Λ' being the complete functions corresponding to this modulus,

$\frac{\Lambda'}{\Lambda} = n\frac{K'}{K}$, so that the equation (6) will be satisfied by assuming $\Sigma = \Theta_{,}(nu)$ or $\Sigma = \mathrm{H}_{,}(nu)$, where $\Theta_{,}$, $\mathrm{H}_{,}$ correspond to the new modulus λ.

Assume now in the equation (6),

$$\Sigma = \left(\frac{\pi}{2}\right)^{\frac{n-1}{2}} (Kk')^{-\frac{n-1}{2}} \Theta^n u.z.$$

Hence, substituting,

$$\frac{d^2}{du^2}(\Theta^n u.z) - 2nu\left(k'^2 - \frac{E}{K}\right)\frac{d}{du}(\Theta^n u.z)$$

$$+ 2nkk'^2 (Kk')^{\frac{n-1}{2}} \frac{d}{dk}\left[(Kk')^{-\frac{n-1}{2}} \Theta^n u.z\right] = 0:$$

but $(Kk')^{\frac{n-1}{2}} \frac{d}{dk}\left[(Kk')^{-\frac{n-1}{2}} \Theta^n u.z\right] = \frac{d}{dk}(\Theta^n u.z) - \frac{n-1}{2Kk'}\frac{dKk'}{dk}\Theta^n u.z,$

or effecting the differentiation, and eliminating $\frac{d\Theta u}{dk}$ by means of the equation obtained from (4) by writing $\Sigma = \Theta u$,

$$(Kk')^{\frac{n-1}{2}} \frac{d}{dk}\left[(Kk')^{-\frac{n-1}{2}} \Theta^n u.z\right]$$

$$= \Theta^n u\left[\frac{dz}{dk} - \frac{nz}{2kk'^2\Theta u}\left\{\frac{d^2\Theta u}{du^2} - 2\left(k'^2 - \frac{E}{K}\right)\frac{d\Theta u}{du}\right\} + \frac{n-1}{2kk'^2}\left(1 - \frac{E}{K}\right)z\right].$$

Substituting in (6) and reducing,

$$\frac{d^2z}{du^2} + 2n\left[\frac{1}{\Theta n}\frac{d\Theta u}{du} - u\left(k'^2 - \frac{E}{K}\right)\right]\frac{dz}{du} + 2nkk'^2\frac{dz}{dk}$$

$$+ n(n-1)\left\{\left[\frac{1}{\Theta^2 u}\left(\frac{d\Theta u}{du}\right)^2 - \frac{1}{\Theta u}\frac{d^2\Theta u}{du^2}\right] + \left(1 - \frac{E}{K}\right)\right\} z = 0,$$

i.e. $$\frac{d^2z}{du^2} + 2n\left[\frac{d\log\Theta u}{du} - u\left(k'^2 - \frac{E}{K}\right)\right]\frac{dz}{du} + 2nkk'^2\frac{dz}{dk}$$

$$+ n(n-1)\left[-\frac{d^2\log\Theta u}{du^2} + \left(1 - \frac{E}{K}\right)\right] z = 0$$

But $$\frac{d\log\Theta u}{du} = u\left(k'^2 - \frac{E}{K}\right) + k^2 \int_0 du\, cn^2 u,$$

$$\frac{d^2\log\Theta u}{du^2} = 1 - \frac{E}{K} - k^2 sn^2 u;$$

whence

$$\frac{d^2z}{du^2}+2nk^2(\textstyle\int_0 du\, cn^2u)\dfrac{dz}{du}+2nkk'^2\dfrac{dz}{dk}+n(n-1)k^2 sn^2u\,.\,z=0 \ldots (7);$$

which is therefore satisfied by

$$z=\left(\frac{2Kk'}{\pi}\right)^{\frac{n-1}{2}}\frac{\Theta_{,}nu}{\Theta^n u},\quad z=\left(\frac{2Kk'}{\pi}\right)^{\frac{n-1}{2}}\frac{\mathrm{H}_{,}nu}{\Theta^n u}:$$

and each of these values are algebraical functions of snu, (viz. either rational functions or rational functions multiplied by $cnu\, dnu$). Also, in the transformation of the n^{th} order,

$$\sqrt{\lambda}\, sn_{,}u=\frac{\mathrm{H}_{,}(nu)}{\Theta_{,}(nu)};$$

so that it is clear that the above values of z may be taken for the denominator and numerator respectively of $\sqrt{\lambda}\, sn_{,}u$; *i.e.* these quantities each of them satisfy the equation (7).

By assuming $\quad x=\sqrt{k}\, sn\, u, \quad a=k+\dfrac{1}{k},$

this becomes

$$n(n-1)x^2z+(n-1)(ax-2x^3)\frac{dz}{dx}$$
$$+(1-ax^2+x^4)\frac{d^2z}{dx^2}-2n(a^2-4)\frac{dz}{da}=0 \ldots (8);$$

which is therefore satisfied by assuming for z either the numerator or the denominator of $\sqrt{\lambda}\, sn_{,}u$ (the transformation of the n^{th} order), which is the form in which the property is given by Jacobi.

In the case where n is odd, the denominator is of the form

$$B_0+B_{,}x^2 \ldots . + B_{\frac{1}{2}(n-1)}x^{n-1},$$

and then the numerator is

$$x(B_{\frac{1}{2}(n-1)} \ldots . + B_{,}x^{n-3}+B_0x^{n-1}),$$

where $\quad B_0=\sqrt{\left(\dfrac{\lambda'}{k'M}\right)},\quad B_{\frac{1}{2}(n-1)}=\sqrt{\left(\dfrac{\lambda\lambda'}{kk'M^3}\right)};$

and all the remaining coefficients may be determined from these, the modular equation being supposed known. But the principal use of the formula is for the multiplication of elliptic functions, which it is well known corresponds to the case where n is a square number. Writing $n=\nu^2$, when ν is odd, the denominator is

$$1+B_2x^4 \ldots . + B_{\frac{1}{2}(\nu^2-3)}x^{\nu^2-3}\pm\nu x^{\nu^2-1},$$

(the $\pm$ sign according as $\nu=(4p+1)$ or $(4p-1)$); and the

numerator is obtained from this by multiplying by x and reversing the order of the coefficients. When ν is even the denominator is

$$1 + B_2 x^4 \dots \pm B_2 x^{\nu^2-4} \pm x^{\nu^2},$$

(+ or −, according as $\nu = 4p$ or $\nu = 4p + 2$), so that there are only half as many coefficients to be determined; but then the numerator must be separately investigated. In general, by leaving (n) indeterminate, and integrating in the form of a series arranged according to ascending powers of x^2; then, whenever n is a square number, the series terminates and gives the denominator of the corresponding formula of multiplication; but the general form of the coefficients has not hitherto been discovered.

By writing $\frac{x}{\sqrt{n}}$ instead of x, and then making n infinite, the equation (8) takes the form

$$x^2 z + ax\frac{dz}{dx} + \frac{d^2z}{dx^2} - 2(a^2 - 4)\frac{dz}{da} = 0 \dots (9):$$

and it is worth while, before attempting the solution of the general case, to discuss this more simple one.*

Assume $z = 1 + C_1 \frac{x^2}{1.2} \dots + C_r \frac{x^{2r}}{1.2 \dots 2r} + \dots;$

then it is easy to obtain

$$C_{r+2} = -(2r+1)(2r+2)C_r - (2r+2)aC_{r+1} + 2(a^2-4)\frac{dC_{r+1}}{da}.$$

The general form may be seen to be

$$C_r = (-)^{r+1}\{2^{2r-3}C_r^1 a^{r-2} + 2^{2r-6}C_r^2 a^{r-4} + \dots\},$$

and then

$$C^p_{r+1} - pC_r^p = -r(2r-1)C_{r-1}^{p-1} + 16(r+2-2p)C_r^{p-1}.$$

The complete value of C_r^p (assuming $C_r^0 = 0$) is given by an equation of the form

$$C_r^p = {}^0C_r^p + {}^1C_r^p\, 2^r + {}^2C_r^p\, 3^r \dots + {}^{p-1}C_r^p\, p^r,$$

---

* Writing $(\beta + 2)$ for a, and putting $z = e^{+\frac{1}{2}x^2}\rho$, this becomes

$$\frac{d^2\rho}{dx^2} - \rho = \beta x^2\rho - \beta x\frac{d\rho}{dx} + (8\beta + 2\beta^2)\frac{d\rho}{d\beta};$$

and if $\rho = \Sigma\, Z_n\beta^n$,

$$\frac{d^2 Z_n}{dx^2} - (8n+1)Z_n = \left(x^2 + 2n - 2 - x\frac{d}{dx}\right)Z_{n-1};$$

from which the successive values of Z_0, Z_1, &c. might be calculated.

where ${}^0C_r^p$, ${}^1C_r^p$, ... are algebraical functions of r of the degrees $2p-2$, $2p-4$, &c. respectively; but as I am not able completely to effect the integration, and my only object is to give an idea of the law of the successive terms, it will be sufficient to consider the first or algebraical term ${}^0C_r^p$, which is determined by the same equation as C_r^p, and moreover completely determined by this equation and the single additional relation $C_r^1 = 1$, since the arbitrary constants of the integration affect only the terms multiplied by 2^r, 3^r, &c.

Assume C_r^p

$$=\frac{1}{[p-1]^{p-1}}\{2^{p-1}L^p[r-2]^{2p-2}+2^{p-2}M^p[r-3]^{2p-3}+\ldots 2^{-p+1}X^p[r-2p]^0\};$$

and substituting this value,

$$\begin{aligned}
&(1-p)\,L^p &&=(1-p)\{L^{p-1} &&\},\\
&(1-p)\,M^p-2p(2-2p)\,L^p &&=(1-p)\{M^{p-1}-11L^{p-1} &&\},\\
&(1-p)\,N^p-2p(3-2p)\,M^p &&=(1-p)\{N^{p-1}-\ \ 7M^{p-1}+12L^{p-1} &&\},\\
&(1-p)\,O^p-2p(4-2p)\,N^p &&=(1-p)\{O^{p-1}-\ \ 3N^{p-1}+30M^{p-1} &&\},\\
&\ \ \vdots && \ \ \vdots
\end{aligned}$$

the law of which is obvious, the coefficients on the second side in the qth line being 1, $4q-19$, and $(2q-3)(2q-2)$ respectively. By successive integrations and substitutions

$$\begin{aligned}
&L^p - L^{p-1} = 0, && L^p = 1,\\
&M^p - M^{p-1} = 4p-11, && M^p = (p-1)(2p-7),\\
&N^p - N^{p-1} = -8p^3+26p^2+49p-114; && \vdots\\
&\ \ \vdots
\end{aligned}$$

(the constants determined by $M^1=0$, $N^1=0$, $O^2=0$, $P^2=0,\ldots$ so as to make C_r^p contain positive powers only of r).

The following are a few of the complete values of C_r^p, the constants determined so as to satisfy $C^p_{p+1}=0$ (except $C_2^1=1$), and the factorials being partially developed in powers of r, viz.

$$\begin{aligned}
C_r^1 &= 1,\\
C_r^2 &= (r-3)(2r-7),\\
C_r^3 &= \tfrac{1}{2}(r-4)(r-5)(4r^2-24r+51),\\
C_r^4 &= \tfrac{1}{6}\{(r+5)(r-6)(r-7)(8r^3-60r^2+286r+63)\\
&\qquad +384\,(9r^2-93.r+242-2.4^r)\},\\
&\text{\&c.}
\end{aligned}$$

(it is curious that C_5^4, C_6^4, C_7^4, all three of them vanish). It seems hopeless to continue this investigation any further.

Returning to the equation (8), and assuming for z an expression of the same form as before, we have, corresponding to the equations before found for the coefficients C_r,

$$C_{r+2} = -(2r+1)(2r+2)(n-2r)(n-2r-1)\,C_r$$
$$-(2r+2)(n-2r-2)\,a\,C_{r+1} + 2n(a^2-4)\frac{dC_{r+1}}{da}.$$

The case corresponding to the denominator in the multiplication of elliptic functions is that of $C_0 = 1$, $C_1 = 0$. It is easy to form the table—

$C_0 = 1,$

$C_1 = 0,$

$C_2 = -2\,n(n-1),$

$C_3 = \;\;8\,n(n-1)(n-4)\,a,$

$C_4 = -4\,n(n-1)(n-4)[n+75] - 32n(n-1)(n-4)(n-9)\,a^2,$

$C_5 = \;\;96\,n(n-1)(n-4)(n-9)[n+44]\,a$
$\quad +128\,n(n-1)(n-4)(n-9)(n-16)\,a^3,$

$C_6 = -24\,n(n-1)(n-4)(n-9)[17n^2+403n+9000]$
$\quad -960\,n(n-1)(n-4)(n-9)(n-16)[n+41]\,a^2$
$\quad -512\,n(n-1)(n-4)(n-9)(n-16)(n-25)\,a^4,$

$C_7 = \;\;96\,n(n-1)(n-4)(n-9)(n-16)[79n^2+2825n+36180]\,a$
$\quad +7168\,n(n-1)(n-4)(n-9)(n-16)(n-25)[n+42]\,a^3$
$\quad +2048\,n(n-1)(n-4)(n-9)(n-16)(n-25)(n-36)\,a^5,$

$C_8 = \;\;48\,n(n-1)(n-4)(n-9)[283n^4 - 26978n^3 + 277827n^2$
$\quad - 5491932n + 127764000]$
$\quad -3840\,n(n-1)(n-4)(n-9)(n-16)(n-25)\times$
$\quad [23n^2+1069n+23436]\,a^2$
$\quad -15360\,n(n-1)(n-4)(n-9)(n-16)(n-25)(n-36)\times$
$\quad [3n+133]\,a^4$
$\quad -8192\,n(n-1)(n-4)(n-9)(n-16)(n-25)(n-36)(n-49)\,a^6,$

&c.

in which of course the coefficient of the highest power of n, in the successive coefficients C_r, is the value of C_r obtained from the equation (8). With regard to the law of these coefficients I have found that

$$C_r = (-)^{r+1}\,2^{2r-3}\,n(n-1^2)\ldots\{n-(r-1)^2\}\,C_r^{\,1}a^{r-2}$$
$$+\,2^{2r-6}\,n(n-1^2)\ldots\{n-(r-2)^2\}\,C_r^{\,2}a^{r-4}$$
$$+\,2^{2r-9}\,n(n-1^2)\ldots\{n-(r-3)^2\}\,C_r^{\,3}a^{r-6}$$
$$+\,\&c.$$

(where however the next term does not contain, as would at first sight be supposed, the factor $n(n-1^2)\ \ldots\ \{n-(r-4)^2\}$.) And then

$$C_r^1 = 1,$$

$$C_r^2 = (r-3)\,[n(2r-7)+(r-1)(8r-7)],$$

$$\begin{aligned} C_r^3 = \tfrac{1}{2}(r-4)(r-5)\,[&n^2(4r^2-24r+51) \\ &+ n(32r^3-220r^2+412r-255) \\ &+ 2(r-1)(r-2)(32r^2-88r+51)]. \end{aligned}$$

In conclusion may be given the following results, in which, recapitulating the notation

$$x = \sqrt{k}\, sn\, u, \quad a = k + \frac{1}{k}, \quad \Delta x = \sqrt{(1 - ax^2 + x^4)},$$

$$\sqrt{k}\, sn\, 2u = \frac{2x\,\Delta x}{1 - x^4},$$

$$\sqrt{k}\, sn\, 3u = \frac{x(3 - 4ax^2 + 6x^4 - x^8)}{1 - 6x^4 + 4ax^6 - 3x^8},$$

$$\sqrt{k}\, sn\, 4u = \frac{4x\,\Delta x\,(1-x^4)(1 - 2ax^2 + 6x^4 - 2ax^6 + x^8)}{1-20x^4+32ax^6-(26+16a^2)x^8+32ax^{10}-20x^{12}+x^{16}},$$

$$\sqrt{k}\, sn\, 5u =$$

$$\frac{x\{5-20ax^2+(62+16a^2)x^4-80ax^6-105x^8+360ax^{10}-(300+240a^2)x^{12} + (368a+64a^3)x^{14}-(125+160a^2)x^{16}+140ax^{18}-50x^{20}+x^{24}\}}{\{1-50x^4+140ax^6-(125+160a^2)x^8+(368a+64a^3)x^{10}-(300+240^2)x^{12} + 360ax^{14}-105x^{16}-80ax^{18}+(62+16a^2)x^{20}-20ax^{22}+5x^{24}\}}$$

&c.

Thus, writing $-x^2$ for x^2, $k=1$, and $\therefore a = 2$,

$$\tan 3u = x\frac{(3+8x^2+6x^4-x^8)}{1-6x^4-8x^6-3x^8} = \frac{x(3-x^2)(1+x^2)^3}{(1-3x^2)(1+x^2)^3} = \frac{x(3-x^2)}{1-3x^2},$$

where $x = \tan u$. (And in general in reducing $\tan nu$ the extraneous factor in the numerator and denominator is $(1+x^2)^{\frac{1}{2}n(n-1)}$.)

58, *Chancery Lane, London, May* 17, 1847.

(To be continued.)

ON CERTAIN ALGEBRAIC FUNCTIONS.

By JAMES COCKLE, M.A., of Trinity College, Cambridge; Barrister-at-Law of the Middle Temple.

I. A HOMOGENEOUS function of the second degree and of m undetermined quantities ξ', ξ'',., $\xi^{(m)}$, may be written as follows:—

$$\kappa_1'^2\xi'^2 + 2(\kappa_1''\xi'' + \kappa_1'''\xi''' + . + \kappa_1^{(m)}\xi^{(m)})\kappa_1'\xi' + f(\xi'', \xi''', ., \xi^{(m)}) ...(1);$$

add to, and subtract from, this expression the square of half the coefficient of $\kappa_1'\xi'$, and let

$$\kappa_1'\xi' + \kappa_1''\xi'' + . + \kappa_1^{(m)}\xi^{(m)} = h_1:$$

then (1) may be put under the form

$$h_1^2 + \phi(\xi'', \xi''', ., \xi^{(m)}).$$

In like manner $\phi(\xi'', \xi''', ., \xi^{(m)})$, which is a homogeneous function of the second degree and of $m-1$ undetermined quantities ξ'', ξ''', ., $\xi^{(m)}$, may be written thus:—

$$\kappa_2''\xi''^2 + 2(\kappa_2'''\xi''' + \kappa_2^{\mathrm{iv}}\xi^{\mathrm{iv}} + . + \kappa_2^{(m)}\xi^{(m)})\kappa_2''\xi'' + \chi(\xi''', \xi^{\mathrm{iv}}, ., \xi^{(m)}),$$

which expression may, by proceeding as before, be reduced to the form

$$h_2^2 + \psi(\xi''', \xi^{\mathrm{iv}}, ., \xi^{(m)}),$$

where
$$h_2 = \kappa_2''\xi'' + \kappa_2'''\xi''' + . + \kappa_2^{(m)}\xi^{(m)};$$

hence (1) may be represented by

$$h_1^2 + h_2^2 + \psi(\xi''', \xi^{\mathrm{iv}}, ., \xi^{(m)});$$

and if we reduce ψ and the corresponding subsequent functions, as we have already done ϕ and the given one, we may put (1) under the form

$$h_1^2 + h_2^2 + . + h_r^2 + . + h_m^2,$$

where
$$h_r = \kappa_r^{(r)}\xi^{(r)} + \kappa_r^{(r+1)}\xi^{(r+1)} + .. \kappa_r^{(m)}\xi^{(m)}.$$

II. A homogeneous function of the third degree and of v undetermined quantities Ξ', ξ', ., $\xi^{(v-1)}$ may be written thus:

$$K'^3\Xi'^3 + 3A'K'^2\Xi'^2 + B'K'\Xi' + C' ...... (2),$$

where A', B', and C' are free from Ξ; and this expression again may be put under the form

$$(K'\Xi' + A')^3 + (B' - 3A'^2)K'\Xi' + C' - A'^3 .... (3).$$

Let
$$K'\Xi' + A' = h_1;$$

then, since $B' - 3A'^2$ is a homogeneous function of the second

degree and of $v-1$ undetermined quantities $\xi', \xi'', ., \xi^{(v-1)}$, we may (by the processes of paragraph I.) put (3) under the form

$$\mathrm{h}_1^3 + (h_1^2 + h_2^2 + . + h^2_{v-1})\, \mathrm{K}'\Xi' + \mathrm{C}' - \mathrm{A}'^3 \ldots . (4);$$

now, the $v-1$ quantities $\xi', \xi'', ., \xi^{(v-1)}$, being perfectly undetermined, we may make

$$h_1^2 + h_2^2 = 0, \quad h_3^2 + h_4^2 = 0, ., .$$

or
$$h_1 + (1)^{\frac{1}{2}} h_2 = 0, \quad h_3 + (1)^{\frac{1}{2}} h_4 = 0, ., \ldots . (5);$$

the last of which lower line of equations is, ($v-1$ being supposed even),

$$h^2_{v-2} + h^2_{v-1} = 0.$$

By means of the system (5) of linear equations, eliminate, from $\mathrm{C}' - \mathrm{A}'^3$, (which is free from Ξ) $v-1$ of the ξ's, and let $f^a(b)$ denote a homogeneous function of the a^{th} degree and of b undetermined quantities; then, since the coefficient of $\mathrm{K}'\Xi'$ has been made to vanish, (4) may now be represented by

$$\mathrm{h}_1^3 + f^2\left(\frac{v-1}{2}\right),$$

for v write v_1 and let $v_{x+1} = \dfrac{v_x - 1}{2}$; then, by processes similar to those which we have just employed, we may reduce the given function of the third degree, to the form

$$\mathrm{h}_1^3 + \mathrm{h}_2^3 + . + \mathrm{h}_x^3 + f^3(v_{x+1}).$$

The equation of finite differences which gives v_x is

$$v_{x+1} - \tfrac{1}{2} v_x = -\tfrac{1}{2},$$

of which the solution is

$$v_x = -1 + \mathrm{C}\left(\tfrac{1}{2}\right)^{x-1} \ldots\ldots\ldots\ldots . (6).$$

Now in order that the given function of the third degree may be reduced to the form of a sum of m cubes, and may, after such reduction, still involve m undetermined quantities, the cubic

$$f^3(v_{m+1}) = 0,$$

to which we shall be conducted when we have arrived at h_m^3, ought to contain *two* undetermined quantities; more than two would be superfluous, hence

$$v_{m+1} = 2 = -1 + \mathrm{C}\left(\tfrac{1}{2}\right)^m \text{ (by 6)},$$

therefore $\quad \mathrm{C} = 3.2^m$, and v_1 (or v) $= 3.2^m - 1$;

also $\quad v_x = 3.2^{m \cdot x+1} - 1$;

and $v_x - 1$ is, of course, even.

III. Adopting a notation employed in the preceding paragraph, let $f^4(u_m)$ denote a homogeneous function of the fourth degree and of u_m undetermined quantities ξ. Then, A', B', C' and D' representing quantities free from ξ', we may (omitting for convenience the multiplier of ξ') make

$$f^4(u_m) = \xi'^4 + 4A'\xi'^3 + B'\xi'^2 + C'\xi' + D'$$
$$= (\xi' + A')^4 + (B' - 6A'^2)\,\xi'^2 + \&c.,$$

which last form of $f^4(u_m)$ we may, continuing and extending the notation of the preceding paragraphs, represent by

$$H_1^4 + f^2(u_m - 1)\,\xi'^2 + f^3(u_m - 1)\,\xi' + f^4(u_m - 1);$$

but, $u_m - 1$ being supposed even, the processes of paragraph I. enable us to make the coefficient of ξ'^2 in this last expression vanish, and, referring to that paragraph, we see that we have the resulting equation

$$f^4(u_m) = H_1^4 + f^3\{\tfrac{1}{2}(u_m - 1)\}\,\xi' + f^4\{\tfrac{1}{2}(u_m - 1)\} \ldots (7).$$

Let u_x be determined from the following equation of finite differences,

$$u_{x+1} = 3.2^{1+2u_x} - 1 \ldots\ldots\ldots\ldots\ldots (8);$$

then, substituting for u_m its value in terms of u_{m-1}, we may change (7) into the following:

$$f^4(u_m) = H_1^4 + f^3(3.2^{2u_{m-1}} - 1)\,\xi' + f^4(3.2^{2u_{m-1}} - 1) \ldots (9),$$

but the result of paragraph II. shews that a homogeneous function of the third degree and of $3.2^{2u_{m-1}} - 1$ undetermined quantities may be reduced to the form of a sum of $2u_{m-1}$ cubes involving $2u_{m-1}$ undetermined quantities. Group these cubes two and two together as we did the squares in (II.) the preceding paragraph, and equate each group to zero. We shall then have a number of equations of the form

$$h_r^3 + h_{r+1}^3 = 0, \quad \text{or} \quad h_r + (1)^{\frac{1}{3}} h_{r+1} = 0,$$

and if by means of these u_{m-1} equations we eliminate from the last term of (9) u_{m-1} of the quantities still remaining undetermined, we shall have, since the coefficient of ξ' will now be zero,

$$f^4(u_m) = H_1^4 + f^4(u_{m-1}),$$

and by similar processes this last equation may be reduced to

$$f^4(u_m) = H_1^4 + H_2^4 + . + H_x^4 + f^4(u_{m-x});$$

hence, in order that $f^4(u_m)$ may be reduced to the form of the sum of m fourth powers and may, after such reduction,

still involve m undetermined quantities, we must, if we wish to have no superfluous quantities to determine, make

$$u_{m-m} = u_0 = 2,$$

and this equation, combined with (8), will completely determine u_m.

IV. So if w_y be determined from the equation

$$w_{y+1} = 3.2^{1+2u_{2w_y}} - 1 \ldots\ldots\ldots\ldots (10),$$

in which u has the same meaning, and is determined in the same manner as before, the method of the preceding paragraph shews that we may at once proceed to the following reduction of $f^5(w_m)$; viz.

$$f^5(w_m) = \mathfrak{h}_1^{\,5} + f^4(u_{2w_{m-1}})\,\xi' + f^5(u_{2w_{m-1}});$$

and if, as the preceding paragraph also shows we may do, we reduce this last equation to the form

$$f^5(w_m) = \mathfrak{h}_1^{\,5} + (\mathrm{H}_1^{\,4} + \mathrm{H}_2^{\,4} + . + \mathrm{H}^4_{2w_{m-1}})\,\xi' + f^5(2w_{m-1}) \ldots (11);$$

and, if we in the same manner as in former cases, by means of w_{m-1} equations of the form

$$\mathrm{H}_r + (1)^{\frac{1}{4}}\,\mathrm{H}_{r+1} = 0,$$

at which we may readily arrive, make the coefficient of ξ' in (11) vanish, the equation (11) will take the form

$$f^5(w_m) = \mathfrak{h}_1^{\,5} + f^5(w_{m-1});$$

which, by proceeding in the same manner, may be further reduced to

$$f^5(w_m) = \mathfrak{h}_1^{\,5} + \mathfrak{h}_2^{\,5} + . + \mathfrak{h}_x^{\,5} + f^5(w_{m-x});$$

and reasoning in the same manner as in the last paragraph, we infer that the equation (10), combined with the following,

$$w_{m-m} = w_0 = 2,$$

will completely determine w_m. This investigation differs from the preceding ones as follows,—the others give absolute results, but this last leaves us an equation of the fifth degree to solve; so that all that we can say is, that we have made the difficulty of reducing $f^5(u_m)$ to the form

$$\mathfrak{h}_1^{\,5} + \mathfrak{h}_2^{\,5} + . + \mathfrak{h}_m^{\,5}$$

(where $\mathfrak{h}_r$ involves $m - r + 1$ undetermined quantities) depend upon that of solving an equation of the fifth degree. The discussion of the equations of differences above given as well as of that which occurs in the succeeding paragraph must be deferred till another opportunity.

V. For u write ${}_4u$, and for w write ${}_5u$, then if ${}_ru_m$ denote the number of disposable quantities necessary in order that an algebraic function of the r^{th} degree of those quantities may be made to satisfy the condition

$$f^r({}_ru_m) = {}^{\backprime}h_1{}^r + {}^{\backprime}h_2{}^r + . + {}^{\backprime}h_m{}^r$$

(${}^{\backprime}h$ having the same meaning as $\mathrm{H}, h, \mathrm{h}, \mathfrak{h}$, &c. and ${}^{\backprime}h_s$ involving $m - s + 1$ undetermined quantities), it will be found that the equation of finite differences

$${}_ru_{x+1} - 3.2^{1 + 2\,{}_4u\; 2\,{}_5u \cdots 2\,{}_ru_x} + 1 = 0,$$

and the subordinate ones implicitly included in it, combined with ${}_ru_0 = 2$, suffice for the determination of ${}_ru_m$, subject to the solution of an equation of the r^{th} degree. This will be seen if we reflect on the preceding paragraphs.

VI. It is hardly necessary to observe that, if m be even, then whenever we can reduce the left hand side of an algebraic equation, of which the right hand side is zero, to the above form, we may, by grouping the h^r's in pairs, equating the sum of each pair to zero, and making an obvious depression of degree, eliminate $\frac{1}{2}m$ of the quantities ξ', &c. between this equation and another, without introducing any elevation of degree by elimination, and without having to solve any equation of a degree higher than the higher of the two given equations.

VII. I now proceed to inquire what is the number of disposable quantities requisite in order that we may, by means of equations whose degrees shall not exceed the r^{th}, simultaneously satisfy a equations of the r^{th} degree, b of the $(r-1)^{\text{th}}$, c of the $(r-2)^{\text{th}}$, . , β of the second, and α of the first degree between those quantities.

VIII. Call the equations of the r^{th} degree the 1st, 2d, and a^{th}, respectively; then, if we can reduce the $(a-1)^{\text{th}}$ equation to the form

$$h_1{}^r + h_2{}^r = 0,$$

the a^{th} equation will be solvable without elevation of degree arising from elimination. Now it will be seen that the processes employed in the reductions here treated of do not in any case conduct to an equation of a degree higher than the r^{th}. So that if we had

$${}_ru_2$$

disposable quantities, we might reduce the solution of the

$(a-1)^{\text{th}}$ and a^{th} equations to that of two equations of the r^{th}, and others of lower degrees.

IX. Again, in order that we may avoid elevation of degree from the $(a-2)^{\text{th}}$ equation, we must reduce it to the sum of $2_r u_2$ powers, and then group the powers two and two and eliminate. But this requires that we should have

$$ {}_r u_{2_r u_2} $$

disposable quantities, and the whole of the equations of the r^{th} degree will require that we should have

$$ {}_r u_{2_r u_{2_r u_{\cdot_{\cdot_{2_r u_2}}}}} $$

disposable quantities; in which expression fully written the letter u would occur $a-1$ times. Call this expression U or $u\left[\begin{smallmatrix} r \\ a \end{smallmatrix}\right]$. Then, from considerations similar to those which have conducted us to the above expression, we find that the b equations of the $(r-1)^{\text{th}}$ degree will increase this last expression to

$$ {}_{r-1} u_{2_{r-1} u_{2_{r-1} u_{\cdot_{\cdot_{2_{r-1} u_U}}}}}; $$

in which expression, when fully written, u would occur b times. Call this last expression

$$ u\begin{bmatrix} r-1, & r \\ b, & a \end{bmatrix}; $$

then these operations must be carried on till we have ascertained how many quantities are required for the reduction of all the equations from the fourth degree upwards. The expression for the number of quantities requisite for this purpose will, on the principle of the above notation, be represented by

$$ u\begin{bmatrix} 4, & 5, & .\,., & r-1, & r \\ \delta, & \epsilon, & .\,., & b, & a \end{bmatrix}. $$

Then, if the equations of the second and third degrees be treated by obvious extensions of processes which I have above given and if the equations of the first degree be also taken into consideration, we shall find that the formula which expresses the number of arbitrary quantities necessary in order that we may make the solution of a equations of

the r^{th} degree, b of the $(r-1)^{\text{th}}$, . , γ of the third, β of the second, and α of the first, between the same unknowns depend upon the equations of the r^{th}, $(r-1)^{\text{th}}$, and lower degrees only, is

$$\alpha + 2^{\beta}\left(3.2^{\,3.2^{\,3.2^{\,3.2^{\cdot^{\cdot^{\cdot^{\,3.2^{\,1+2u\left[\begin{matrix}4, & 5, & ., & r-1, & r\\ \delta, & \epsilon, & ., & b, & a\end{matrix}\right]}}}}}-1}-1}-2}-1\right),$$

or Υ;

in which formula there are supposed to be γ lines of exponents, γ being the number of equations of the third degree which it is required to satisfy. Were these formulæ incapable of giving illusory results, there would remain but little to be done in the Theory of Algebraic Equations; the formulæ would also have other applications. But in order to ascertain the limits of their application, and to compare particular cases of them with corresponding cases in Mr. Jerrard's method, we must ascertain the results which follow from supposing $\Upsilon - v$ of the quantities ξ', ξ'', &c. to be functions of the remaining v of them. This point I shall defer, but I hope to discuss the question at some future time in its most general form and so as to include the isolated results at which I have already arrived in a contemporary periodical.*

* This paper contains, I think, everything necessary for the complete development of the method of which I have already given isolated discussions in the *Philosophical Magazine*, and the *Mathematician*.

Postscript. June 14, 1847. The reduction which I suggested in a short note, published at pages 285–286 of vol. I. of the present series of this work, will be found to simplify my discussion at page 105 of vol. III. of the former series.

2, *Church-Yard Court, Temple,*
February 25, 1847.

* The *London, Edinburgh, and Dublin Philosophical Magazine.*

ON CONJUGATE HYPERBOLOIDS.

By Thomas Weddle.

If
$$\left.\begin{array}{ll} l_1^2 + m_1^2 - n_1^2 = 1, & l_1 l_2 + m_1 m_2 - n_1 n_2 = 0 \\ l_2^2 + m_2^2 - n_2^2 = 1, & l_1 l_3 + m_1 m_3 - n_1 n_3 = 0 \\ l_3^2 + m_3^2 - n_3^2 = -1, & l_2 l_3 + m_2 m_3 - n_2 n_3 = 0 \end{array}\right\} \ldots (\alpha),$$

then will
$$\left.\begin{array}{ll} l_1^2 + l_2^2 - l_3^2 = 1, & l_1 m_1 + l_2 m_2 - l_3 m_3 = 0 \\ m_1^2 + m_2^2 - m_3^2 = 1, & l_1 n_1 + l_2 n_2 - l_3 n_3 = 0 \\ n_1^2 + n_2^2 - n_3^2 = -1, & m_1 n_1 + m_2 n_2 - m_3 n_3 = 0 \end{array}\right\} \ldots (\beta);$$

and
$$\left.\begin{array}{ll} \pm l_1 = m_2 n_3 - m_3 n_2, & \pm l_2 = m_3 n_1 - m_1 n_3, \\ \pm m_1 = l_3 n_2 - l_2 n_3, & \pm m_2 = l_1 n_3 - l_3 n_1, \\ \mp n_1 = l_2 m_3 - l_3 m_2, & \mp n_2 = l_3 m_1 - l_1 m_3, \\ \multicolumn{2}{c}{\mp l_3 = m_1 n_2 - m_2 n_1} \\ \multicolumn{2}{c}{\mp m_3 = l_2 n_1 - l_1 n_2} \\ \multicolumn{2}{c}{\pm n_3 = l_1 m_2 - l_2 m_1} \end{array}\right\} \ldots (\gamma).$$

That (β) and (γ) are necessary consequences of (α) is easily seen by substituting $-n_1\sqrt{(-1)}$, $-n_2\sqrt{(-1)}$, $l_3\sqrt{(-1)}$ and $m_3\sqrt{(-1)}$ for n_1, n_2, l_3 and m_3 in the equations (A), (B), (C) given at p. 13, No. vii. of this Journal (Jan. 1847).

Def. An hyperboloid of one sheet and an hyperboloid of two sheets are denominated *conjugate hyperboloids* when the real and imaginary axes of the one coincide (both in position and magnitude) with the imaginary and real axes of the other.

Hence the hyperboloid of one sheet denoted by

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{z^2}{c^2} = 1 \ldots\ldots\ldots\ldots (1),$$

and that of two sheets denoted by

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{z^2}{c^2} = -1 \ldots\ldots\ldots\ldots (2)$$

are conjugate hyperboloids; also, these surfaces have evidently the same asymptotic cone, the equation of which is

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{z^2}{c^2} = 0 \ldots\ldots\ldots\ldots (3).$$

Let (x_1, y_1, z_1), (x_2, y_2, z_2), and (x_3, y_3, z_3) be three conjugate points on the conjugate hyperboloids (1, 2); and since two of these points must be on the hyperboloid of one sheet, and one on that of two sheets, let (x_3, y_3, z_3) be the point on the latter. The equations to the conjugate tangent planes touching at these points will be

$$\frac{x_1 x}{a^2} + \frac{y_1 y}{b^2} - \frac{z_1 z}{c^2} = 1 \ldots\ldots\ldots\ldots (4),$$

$$\frac{x_2 x}{a^2} + \frac{y_2 y}{b^2} - \frac{z_2 z}{c^2} = 1 \ldots\ldots\ldots\ldots (5),$$

and
$$\frac{x_3 x}{a^2} + \frac{y_3 y}{b^2} - \frac{z_3 z}{c^3} = -1 \ldots\ldots\ldots\ldots (6).$$

The following relations (7, 8, 9) may be established exactly as those marked (5, 6, 7) were at p. 14, No. VII.:

$$\frac{x_1 x_2}{a^2} + \frac{y_1 y_2}{b^2} - \frac{z_1 z_2}{c^2} = 0 \ldots\ldots\ldots\ldots (7),$$

$$\frac{x_1 x_3}{a^2} + \frac{y_1 y_3}{b^2} - \frac{z_1 z_3}{c^2} = 0 \ldots\ldots\ldots\ldots (8),$$

and
$$\frac{x_2 x_3}{a^2} + \frac{y_2 y_3}{b^2} - \frac{z_2 z_3}{c^2} = 0 \ldots\ldots\ldots\ldots (9).$$

Also, since the conjugate points are on the hyperboloids, we have (1, 2),

$$\frac{x_1^2}{a^2} + \frac{y_1^2}{b^2} - \frac{z_1^2}{c^2} = 1 \ldots\ldots\ldots\ldots (10),$$

$$\frac{x_2^2}{a^2} + \frac{y_2^2}{b^2} - \frac{z_2^2}{c^2} = 1 \ldots\ldots\ldots\ldots (11)$$

$$\frac{x_3^2}{a^2} + \frac{y_3^2}{b^2} - \frac{z_3^2}{c^2} = -1 \ldots\ldots\ldots\ldots (12).$$

Now, if $\frac{x_1}{a}$, $\frac{y_1}{b}$, $\frac{z_1}{c}$, $\frac{x_2}{a}$, &c. be written in (α) for l_1, m_1, n_1, l_2, &c., the resulting equations will be true by (10, 11, 12, 7, 8, 9); hence, making the same substitutions in (β) and (γ), we have

$$x_1^2 + x_2^2 - x_3^2 = a^2 \ldots\ldots\ldots\ldots (13),$$

$$y_1^2 + y_2^2 - y_3^2 = b^2 \ldots\ldots\ldots\ldots (14).$$

$$z_1^2 + z_2^2 - z_3^2 = -c^2 \ldots\ldots\ldots\ldots (15).$$

$$x_1 y_1 + x_2 y_2 - x_3 y_3 = 0 \ldots\ldots\ldots\ldots (16),$$

$$x_1 z_1 + x_2 z_2 - x_3 z_3 = 0 \ldots\ldots\ldots\ldots (17),$$

$$y_1 z_1 + y_2 z_2 - y_3 z_3 = 0 \ldots\ldots\ldots\ldots (18).$$

$$\pm \frac{x_1}{a} = \frac{y_2 z_3 - y_3 z_2}{bc}, \quad \pm \frac{x_2}{a} = \frac{y_3 z_1 - y_1 z_3}{bc}, \quad \mp \frac{x_3}{a} = \frac{y_1 z_2 - y_2 z_1}{bc} \ldots (19),$$

$$\pm \frac{y_1}{b} = \frac{x_3 z_2 - x_2 z_3}{ac}, \quad \pm \frac{y_2}{b} = \frac{x_1 z_3 - x_3 z_1}{ac}, \quad \mp \frac{y_3}{b} = \frac{x_2 z_1 - x_1 z_2}{ac} \ldots (20),$$

$$\mp \frac{z_1}{c} = \frac{x_2 y_3 - x_3 y_2}{ab}, \quad \mp \frac{z_2}{c} = \frac{x_3 y_1 - x_1 y_3}{ab}, \quad \pm \frac{z_3}{c} = \frac{x_1 y_2 - x_2 y_1}{ab} \ldots (21).$$

The preceding equations are analogous to those given in my paper entitled "Investigation of certain Properties of the Ellipsoid," (see *Journal*, New Series, vol. II., pp. 13–19), of which communication this is designed to form a continuation. Most of the theorems deduced in that paper are true of conjugate hyperboloids with some slight modifications, the chief of which arise from the squares of all the lines that refer to the hyperboloid of two sheets (those lines which have 3 subscribed) having a negative sign.

As the investigations of the following properties of conjugate hyperboloids, (a). . . .(g), are the same as those of the analogous ones (A). . . .(G), in the paper just mentioned, I shall, for brevity's sake, omit them; also since the enunciations of (b) and (c) would be the same as those of (B) and (C), the latter will, for the same reason, be simply referred to.

(a) If three conjugate points be projected on any diametral plane by lines drawn parallel to the diameter conjugate to this plane, the difference between the square of the line of projection drawn from the point on the hyperboloid of two sheets, and the sum of the squares of the other two lines, is equal to the square of the semidiameter.

(b) The same as (B), see *Journal*, vol. II. p. 15, New Series.

(c) The same as (C), *ib.*

(d) The difference between the square of any diameter of the hyperboloid of two sheets, and the sum of the squares of two conjugate diameters (which appertain to the hyperboloid of one sheet) is constant. Hence, should it happen that $a^2 + b^2 = c^2$, the square of any diameter of the hyperboloid of two sheets will be equal to the sum of the squares of two conjugate diameters.

(e) Each conjugate parallelepiped* is equal to that constructed on the principal diameters.

(f) The difference between the sum of the squares of those two faces of any conjugate parallelepiped which touch the hyperboloid of two sheets, and the sum of the squares of the other four faces (which touch the hyperboloid of one sheet) is constant. Hence, if $a^2b^2 = a^2c^2 + b^2c^2$ or $\frac{1}{c^2} = \frac{1}{a^2} + \frac{1}{b^2}$, the sum of the squares of the two faces will be equal to the sum of the squares of the four.

(g) If perpendiculars be drawn from the centre of conjugate hyperboloids on any three conjugate tangent planes, the difference between the square of the reciprocal of the perpendicular on the tangent plane to the hyperboloid of two sheets, and the sum of the squares of the reciprocals of the other two perpendiculars, is constant. Hence if $\frac{1}{c^2} = \frac{1}{a^2} + \frac{1}{b^2}$, the square of the reciprocal of the former perpendicular will be equal to the sum of the squares of the reciprocals of the other two.

The locus of the intersections of three conjugate tangent planes will be obtained by eliminating x_1, x_2, &c. from (4, 5, 6), and this elimination is at once effected by deducting the square of (6) from the sum of the squares of (4, 5), and reducing by (13)....(18); therefore

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{z^2}{c^2} = 1.$$

Hence

(h) The locus of the intersections of conjugate tangent planes to conjugate hyperboloids is the hyperboloid of one sheet itself. Hence, also,

(i) Every conjugate parallelepiped to conjugate hyperboloids is inscribed in the hyperboloid of one sheet itself.

The theorems (h, i) which differ widely from the corresponding propositions (H, I) for the ellipsoid, are very remarkable, and, I must confess, these results were totally unexpected by me. Recollecting that conjugate parallelepipeds to conjugate hyperboloids are in some respects analogous to conjugate parallelograms to conjugate hyperbolas,

* The faces of a *conjugate parallelepiped* touch the conjugate hyperboloids and are parallel to conjugate diametral planes.

I imagined that the locus would be the asymptotic cone (3). It is not, however, conjugate parallelepipeds, but conjugate cylinders (to be noticed presently) that are here analogous to conjugate parallelograms.

Theorems in reference to conjugate hyperboloids have now been given analogous to all those contained in the paper on the ellipsoid, except to (K), (L), and (M), and I have not been able to discover any properties similar to these. (In consequence none of the following propositions will be marked (k), (l), or (m).) I shall now introduce a few additional properties of the hyperboloids, several of which have analogues in Plane Geometry.

Referring the conjugate hyperboloids (1, 2) and the asymptotic cone (3) to any conjugate diameters, $A_1'OA_1 = 2a_1$, $B_1'OB_1 = 2b_1$, and $C_1'OC_1 = 2c_1$, their equations will be

$$\frac{x^2}{a_1^2} + \frac{y^2}{b_1^2} - \frac{z^2}{c_1^2} = 1 \ldots\ldots\ldots\ldots\ldots (22),$$

$$\frac{x^2}{a_1^2} + \frac{y^2}{b_1^2} - \frac{z^2}{c_1^2} = -1 \ldots\ldots\ldots\ldots (23),$$

and

$$\frac{x^2}{a_1^2} + \frac{y^2}{b_1^2} - \frac{z^2}{c_1^2} = 0 \ldots\ldots\ldots\ldots\ldots (24).$$

The equations of the tangent planes at C_1' and C_1 to the hyperboloid of two sheets (23), are $z = -c_1$, and $z = c_1$; and either of these values of z substituted in (24) gives $\frac{x^2}{a_1^2} + \frac{y^2}{b_1^2} = 1$, for the equation to the section of the cone by the tangent plane at C_1' or C_1; now this is the very same equation we should get by putting $z = 0$ in (22); hence

(n) If sections of the asymptotic cone be made by two parallel tangent planes to the hyperboloid of two sheets, each section (which is an ellipse) is equal, similar and similarly posited, to the section of the conjugate hyperboloid made by a parallel diametral plane.

Again, let sections of (22, 23, 24) be made by the plane $z = z_1$, parallel to that of xy. The areas of these sections being denoted by L, M, N, respectively, and the angle A_1OB_1 by ϕ, we shall evidently have

$$\text{L} = \pi a_1 b_1 \left(\frac{z_1^2}{c_1^2} + 1\right) \sin\phi, \qquad \text{M} = \pi a_1 b_1 \left(\frac{z_1^2}{c_1^2} - 1\right) \sin\phi,$$

$$\text{N} = \pi a_1 b_1 \frac{z_1^2}{c_1^2} \sin\phi;$$

therefore $L - N = N - M = \pi a_1 b_1 \sin \phi$ = area of parallel diametral section. It is hence easily seen that

(o) If elliptic sections of two conjugate hyperboloids and the asymptotic cone be made by any plane, the area of each of the two elliptic rings bounded by the curves of section will be equal to that of the parallel diametral section, or, (n), to that of the section of the cone made by a parallel tangent plane. Hence also the elliptic rings are equal in area for all parallel sections.

Def. *A conjugate cylinder* to conjugate hyperboloids has its generators parallel to a diameter of the hyperboloid of two sheets and tangent to the hyperboloid of one sheet, and it is limited by the tangent planes touching at the extremities of the diameter.

It is plain that

(p) The diametral section parallel to the ends of a conjugate cylinder is the locus of the points of contact of the generators with the hyperboloid of one sheet, and hence also, (n), the asymptotic cone is the locus of the perimeters of the ends of all conjugate cylinders.

It is evident that a conjugate cylinder and the corresponding conjugate parallelepiped have the same altitude, and that their bases are in the proportion $\pi : 4$; hence, (e),

(q) All conjugate cylinders are equal to each other, and the volume of each is $2\pi abc$; a, b, c being the principal semi-diameters.

Moreover, the portion of the asymptotic cone cut off by a tangent plane to the hyperboloid of two sheets has evidently the same base as the corresponding conjugate cylinder, but only half its altitude; the volume of the former solid is consequently equal to one-sixth of that of the latter. Hence the following remarkable theorem.

(r) The volume of the portion of the asymptotic cone cut off by a tangent plane to an hyperboloid of two sheets is constant and equal to $\frac{1}{3}\pi abc$.

I shall next establish the following theorem.

(s) If from any point in an hyperboloid of two sheets as vertex, a cone be described having its generators parallel to those of the asymptotic cone, the volume of the solid included between the surfaces of these two cones is constant and equal to $\frac{1}{12}\pi abc$.

For the tangent plane at the point will, (r), cut off from the asymptotic cone a solid $U = \frac{1}{3}\pi abc$, and the solid V mentioned in (s) is evidently composed of two solids W similar to U, but of only half the (linear) dimensions; hence $W = \frac{1}{8}U = \frac{1}{24}\pi abc$, and $V = 2W = \frac{1}{12}\pi abc$.

(t) If any straight line be drawn cutting an hyperboloid in U_1, V_1, the conjugate hyperboloid in U_2, V_2, and the asymptotic cone in U, V; then will $UU_1 = VV_1$, $UU_2 = VV_2$, and $UU_1 . U_1V = UV_1 . V_1V = UU_2 . U_2V = UV_2 . V_2V$ = square of the parallel semidiameter.

The truth of (t) may easily be shewn by referring the surfaces to conjugate diameters, one of which shall be parallel to the straight line. It will also be apparent by drawing a diametral plane through the straight line which will cut the hyperboloids in conjugate hyperbolas and the asymptotic cone in their asymptotes; then, applying well-known properties of the hyperbola, we shall have the theorems (t) at once.

In conclusion, I would observe that the consideration of conjugate hyperboloids seems to be as necessary as that of conjugate hyperbolas. We can obtain a clear geometrical conception of many theorems when enunciated as properties of conjugate hyperboloids, of which we have, I think, but an obscure notion when presented as properties of only *one* of these surfaces. In truth, the few properties of the kind here alluded to, that are usually given in works on Analytical Geometry of Three Dimensions, are enunciated with a tacit reference to the ellipsoid, and the student is afterwards merely informed that the squares of certain quantities are negative in the case of either of the hyperboloids. He is thus furnished with analytical expressions, but with only a very confused idea of their geometrical meaning. The introduction of the conjugate hyperboloid, however, completely dispels this obscurity, and enables us to enunciate such theorems with precision.

ADDENDUM. Since the preceding paper was sketched, it has occurred to me that most of the theorems, (o)....(t), have analogues in reference to the ellipsoid, while some of them are capable of being enunciated with still greater generality. I shall insert these propositions here, but shall omit the investigations (which indeed are not difficult) in order to save space.

DEF. A *conjugate cylinder* circumscribed about an ellipsoid is a cylinder whose ends touch the ellipsoid at the extremities of the diameter parallel to the generators.

(P) The locus of the perimeters of the ends of conjugate cylinders circumscribed about an ellipsoid is a concentric similar ellipsoid whose principal diameters are to those of the given ellipsoid as $\sqrt{2} : 1$.

(Q) An ellipsoid is two-thirds of each circumscribed conjugate cylinder, and hence all conjugate cylinders circumscribed about the same ellipsoid are equal to one another.

The former part of this proposition is an extension of a property of the sphere.

(O) If elliptic sections of two concentric, similar, and similarly situated surfaces of the second order be made by parallel planes, the elliptic rings bounded by the curves of section will be equal to each other.

(R) Tangent planes to the inner of two concentric, similar, and similarly posited surfaces of the second order cut off equal volumes from the other.

The propositions (O) and (R) hold if the surfaces are either ellipsoids or hyperboloids, and there are analogous properties in respect of two *equal* elliptic paraboloids which have their principal axes in the same straight line and are similarly posited.* The following properties, (T), are true of any of the surfaces of the second order, providing the enunciation be modified as before for the paraboloids.

(T) If there be two concentric, similar, and similarly posited surfaces of the second order and any straight line be drawn cutting the outer surface in U', V', and the other in U'', V''; then $U'U'' = V'V''$, and $U'U''.U''V' = U'V''.V''V'$ is constant for all parallel lines. (When the surfaces are ellipsoids and are related to each other as in (P), $U'U''.U''V' = U'V''.V''V' =$ square of the parallel semidiameter of the inner ellipsoid.)

To be able to perceive that (O), (R), and (T) are extensions of (o), (r), and (t), it must be recollected that a cone is a limiting case of either of the hyperboloids.

Cottenham St., Newcastle-upon-Tyne,
May 4, 1847.

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* That is, providing the equations to the two paraboloids be

$$z + d = \frac{x^2}{p_1} + \frac{y^2}{p_2}, \text{ and } z + d' = \frac{x^2}{p_1} + \frac{y^2}{p_2},$$

the principal axis of each being the axis of z.

NOTES ON HYDRODYNAMICS.

I.—*On the Equation of Continuity.*

By William Thomson.

The following proof of the Equation of Continuity is simpler than that which is generally given in treatises on Hydrodynamics, and it has also the advantage of shewing in a clearer manner the nature of the property of fluid motion expressed.* Thus, instead of considering a portion of the moving fluid and the varying space which the particles composing it occupy at successive instants, as in the ordinary proof, we imagine a space S fixed in the interior of the fluid, and we consider the fluid which flows into this space, across part of the bounding surface, and that which flows out of it, across the remainder in a given interval of time. The equation of continuity is the analytical expression of the fact that the change in the mean density of the fluid in the space S, during the interval of time considered, is due to the difference between the quantities of fluid which, in that interval, flow into it and out of it, or, if the fluid be of invariable density, that these quantities are equal; and its generality, as applied to all cases of fluid motion, is subject to no exception.

Let the space S be an infinitely small parallelepiped, of which the edges α, β, γ are parallel to the axes of coordinates, and let x, y, z be the coordinates of its centre; so that $x \pm \frac{1}{2}\alpha$, $y \pm \frac{1}{2}\beta$, $z \pm \frac{1}{2}\gamma$ are the coordinates of its angular points. Let ρ be the density of the fluid at (x, y, z), or the mean density through the space S, at the time t. The density at the time $t + dt$ will be $\rho + \frac{d\rho}{dt}dt$; and hence the quantities of fluid contained in the space S, at the times t, and $t + dt$, are respectively $\rho \,.\, \alpha\beta\gamma$ and $\left(\rho + \frac{d\rho}{dt}dt\right)\alpha\beta\gamma$. Hence the quantity of fluid lost (there will of course be an absolute gain if $\frac{d\rho}{dt}$ be positive) in the time dt is

$$-\frac{d\rho}{dt}dt \,.\, \alpha\beta\gamma \quad \ldots\ldots\ldots\ldots\ldots (a).$$

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* Poisson admits that the proof which he gives is inapplicable to certain conceivable circumstances of fluid motion; but he erroneously concludes that in such cases the equation "of continuity" does not hold. (See Poisson's *Traité de Mécanique*, No. 651.) The proof in the text has been frequently given in lectures at Cambridge, and elsewhere, and it is likely to occur to any one reading Fourier's Theory of Heat; but I am not aware that it has been hitherto published in any work except Duhamel's *Cours de Mécanique* (Deuxième Partie; Paris 1847).

Now let u, v, w be the three components of the velocity of the fluid (or of a fluid particle*) at P. These quantities will be functions of x, y, z, (involving also t, except in the case of "steady motion,") and will in general vary gradually from point to point of the fluid; although the analysis which follows is not restricted by this consideration, but holds even in cases where in certain places of the fluid there are abrupt transitions in the velocity, as may be seen by considering them as limiting cases of motions in which there are very sudden continuous transitions of velocity. If ω be a small plane area, perpendicular to the axis of x, and having its centre of gravity at P, the volume of fluid which flows across it in the time dt will be equal to $u \,.\, \omega \,.\, dt$, and the mass or quantity will be $\rho \,.\, u \,.\, \omega \,.\, dt$. If we substitute $\beta\gamma$ for ω, the quantity which flows across the either of the sides $\beta\gamma$ of the parallelepiped S, will differ from this only on account of the variation in the value of ρu; and therefore the quantities which flow across the two sides $\beta\gamma$ are respectively

$$\left\{\rho u - \tfrac{1}{2} a \frac{d(\rho u)}{dx}\right\} . \beta\gamma \,.\, dt,$$

and

$$\left\{\rho u + \tfrac{1}{2} a \frac{d(\rho u)}{dx}\right\} . \beta\gamma \,.\, dt.$$

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* This explanatory clause must be omitted, and a modified definition of fluid velocity must be given, if it be required to include the case, imagined by Poisson, of the motion of two fluids of different densities *through* one another, in which, as he conceives, the "*molécules*" of the lighter fluid will move upwards, between the "*molécules*" of the heavier fluid, which descend. Thus we should define u as the mean velocity of the "*molécules*" parallel to the axis of x, across any very small plane of which the centre of gravity is at P, and we should thus obtain the same equation as that found in the text, which is applicable to this as to every possible case of fluid motion. It is however very doubtful whether this kind of motion can actually exist in nature. In the case, considered by Poisson (Art. 651), of water contained in a vertical cylinder, open above, and heated at its bottom which is supposed to be horizontal, it is certainly true that the regular upward motion of the whole fluid due to the expansion of the lower strata is practically impossible, because unstable; but as far as experience indicates, (by the *streaks* we can see on looking into the vessel, on account of the varying refracting power of the heterogeneous liquid,) we find that the effect of the instability is to disturb the surfaces of equal density from being horizontal planes, and thus to allow finite portions of the lighter fluid to ascend, their places being filled by the heavier fluid descending. The definition, in the text, of the components, u, v, w, is directly applicable to this kind of motion; and the analysis and resulting equation, when interpreted according to the principles of the differential calculus as applied to discontinuous functions, will not be subject to exception even in cases when the ascending and descending portions slide upon one another with finite velocities; cases which might actually occur were there no "friction of fluids in motion."

Hence $\alpha \dfrac{d(\rho u)}{dx} . \beta\gamma . dt$, or $\dfrac{d(\rho u)}{dx} . \alpha\beta\gamma . dt$ is the excess of the quantity of fluid which leaves the parallelepiped across one of the faces $\beta\gamma$ above that which enters it across the other. By considering in addition the effect of the motion across the other faces of the parallelepiped, we find for the total quantity of fluid lost from the space S, in the time dt,

$$\left\{\frac{d(\rho u)}{dx} + \frac{d(\rho v)}{dy} + \frac{d(\rho w)}{dz}\right\} . \alpha\beta\gamma . dt \,\ldots\ldots (b).$$

Equating this to the expression (a), previously found, we have

$$\left\{\frac{d(\rho u)}{dx} + \frac{d(\rho v)}{dy} + \frac{d(\rho w)}{dz}\right\} . \alpha\beta\gamma . dt = -\frac{d\rho}{dt} . dt . \alpha\beta\gamma;$$

and we deduce

$$\frac{d(\rho u)}{dx} + \frac{d(\rho v)}{dy} + \frac{d(\rho w)}{dz} + \frac{d\rho}{dt} = 0 \,\ldots\ldots\ldots (1),$$

which is the required equation.

If, instead of taking the infinitely small parallelepiped $\alpha\beta\gamma$, and the infinitely small interval of time dt, we consider a finite space S bounded by a fixed surface, and a finite interval of time, from t_1 to t_2, the equation of continuity should, it is clear from the demonstration given above, express this fact; that the mass of fluid in S at the time t_2 is equal to the mass at the time t_1, wanting the total mass which has been taken away by the flux across the surface. This is verified directly by the following analytical process.*

Let ρ_1, ρ_2 be the densities of the fluid at (xyz) at the times t_1, t_2, and let M_1, M_2 be the total masses contained at those times in the space S. We shall have

$$M_1 = \iiint \rho_1 \, dx\, dy\, dz, \quad M_2 = \iiint \rho_2 \, dx\, dy\, dz,$$

and therefore $M_2 = M_1 + \iiint (\rho_2 - \rho_1)\, dx\, dy\, dz$

$$= M_1 + \int_{t_1}^{t_2} dt \iiint \frac{d\rho}{dt}\, dx\, dy\, dz \,\ldots\ldots\ldots (2).$$

But, by the equation of continuity, we find

$$-\iiint \frac{d\rho}{dt}\, dx\, dy\, dz = \iiint \left\{\frac{d(\rho u)}{dx} + \frac{d(\rho v)}{dy} + \frac{d(\rho w)}{dz}\right\} dx\, dy\, dz,$$

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* Compare *Camb. Math. Jour.* vol. III. p. 203; also Poisson, *Théorie de la Chaleur*, p. 177.

and hence, by separating the second member into three terms, and performing the integrations, in the first with respect to x, in the second with respect to y, and in the third with respect to z, and assigning the limits so as to include the whole space S, we have

$$-\iiint \frac{d\rho}{dt}\, dx\, dy\, dz = \iint \rho u . dy\, dz + \iint \rho v . dz\, dx + \iint \rho w . dx\, dy \ldots (3),$$

where the values of xyz in each term of the second member belong to the surface of S. Now let ds be an element of the surface at xyz, and let l, m, n be the direction cosines of a normal: we may take ds such that

$$ds . l = dy\, dz, \quad ds . m = dz\, dx, \quad ds . n = dx\, dy\,;$$

and modifying accordingly the second member of (3), we have

$$-\iiint \frac{d\rho}{dt}\, dx\, dy\, dz = \iint \rho (lu + mv + nw)\, ds \ldots (4).$$

Hence (2) becomes

$$M_2 = M_1 - \int_{t_1}^{t_2} dt \iint \rho (lu + mv + nw)\, ds,$$

or

$$M_2 = M_1 - \iint ds . \int_{t_1}^{t_2} \rho (lu + mv + nw)\, dt \ldots\ldots (5).$$

Now $lu + mv + nw$ is the component of the velocity of the fluid in the direction of the normal at xyx, and therefore $ds . \int_{t_1}^{t_2} \rho (lu + mv + nw)\, dt$ is the quantity which flows out of the space S, across the element ds, of the surface, in the interval considered. Hence the total quantity, lost from S in the time $t_2 - t_1$, is equal to the integral in the second member of equation (5), and this equation is therefore the expression of the required result.

If the mass we are considering be a liquid (that is to say, an incompressible fluid), even although it be heterogeneous, the equation of continuity assumes a simpler form. For the density at a point xyz, moving with the fluid, will be invariable, and therefore the differential of ρ, considered as a function of x, y, z, t, will vanish provided we take $dx = udt$, $dy = vdt$, $dz = wdt$. Hence

$$\frac{d\rho}{dt} dt + \frac{d\rho}{dx} . udt + \frac{d\rho}{dy} . vdt + \frac{d\rho}{dz} . wdt = 0.$$

Dividing by dt, then subtracting the first member from that of the equation of continuity, and dividing the result by ρ, we find

$$\frac{du}{dx} + \frac{dv}{dy} + \frac{dw}{dz} = 0 \text{ .... } (6).$$

This equation has the same form as in the case when the liquid is homogeneous, as might easily have been proved directly, by considering merely the volume of fluid which flows through the space S, and not its mass, which, when the fluid is incompressible, will enable us to arrive at the equation of continuity.

St. Peter's College, Sept. 29, 1847.

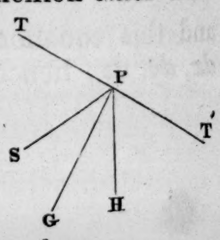
MATHEMATICAL NOTE.

On the Maximum or Minimum Property of Incident and Reflected Rays.

THE following is a very simple analytical proof of the proposition, that when a ray of light is reflected at any surface, the length of the path of the ray, measured from a given point in the incident to a given point in the reflected ray, is less than it would be according to any law of reflexion other than the actual law.

Let P be the point of incidence, SP the incident, PH the reflected ray, PG the normal to the surface at P.

Then we have to prove, that if $SP + PH$ is a minimum,



then (1) SP, PH, PG are in the same plane.

(2) $SPG = HPG$.

Let $x\,y\,z$ be the coordinates of P,

$\alpha\,\beta\,\gamma$ .... S,

$\alpha'\beta'\gamma'$ .... H.

$SP = r$, $PH = r'$. Then the condition that $r + r'$ shall be a minimum gives us

$$dr + dr' = 0,$$

$$\text{or } \frac{dr}{dx}dx + \frac{dr}{dy}dy + \frac{dr}{dz}dz + \frac{dr'}{dx}dx + \frac{dr'}{dy}dy + \frac{dr'}{dz}dz = 0 \text{...(A).}$$

Now it will be easily seen that the equations of *SP, PH, PG*, are respectively

$$\frac{x_1 - x}{\frac{dr}{dx}} = \frac{y_1 - y}{\frac{dr}{dy}} = \frac{z_1 - z}{\frac{dr}{dz}},$$

$$\frac{x_1 - x}{\frac{dr'}{dx}} = \frac{y_1 - y}{\frac{dr'}{dy}} = \frac{z_1 - z}{\frac{dr'}{dz}},$$

$$\frac{x_1 - x}{\frac{df}{dx}} = \frac{y_1 - y}{\frac{df}{dy}} = \frac{z_1 - z}{\frac{df}{dz}},$$

f being such a function that $f(x, y, z) = 0$ is the equation of the given surface. In order that these may lie in the same plane, we must have

$$\frac{df}{dx}\left(\frac{dr}{dy}\frac{dr'}{dz} - \frac{dr}{dz}\frac{dr'}{dy}\right) + \frac{df}{dy}\left(\frac{dr}{dz}\frac{dr'}{dx} - \frac{dr}{dx}\frac{dr'}{dz}\right)$$
$$+ \frac{df}{dz}\left(\frac{dr}{dx}\frac{dr'}{dy} - \frac{dr}{dy}\frac{dr'}{dx}\right) = 0 \ldots \text{(B)}.$$

But $$\frac{df}{dx}dx + \frac{df}{dy}dy + \frac{df}{dz}dz = 0,$$

and this equation and (A) are the only relations between dx, dy, dz; hence we may take

$$\frac{df}{dx} = \lambda\left(\frac{dr}{dx} + \frac{dr'}{dx}\right),$$

$$\frac{df}{dy} = \lambda\left(\frac{dr}{dy} + \frac{dr'}{dy}\right),$$

$$\frac{df}{dz} = \lambda\left(\frac{dr}{dz} + \frac{dr'}{dz}\right).$$

If we substitute these values, (B) assumes the form

$$\left(\frac{dr}{dx} + \frac{dr'}{dx}\right)\left(\frac{dr}{dy}\frac{dr'}{dz} - \frac{dr}{dz}\frac{dr'}{dy}\right) + \left(\frac{dr}{dy} + \frac{dr'}{dy}\right)\left(\frac{dr}{dz}\frac{dr'}{dx} - \frac{dr}{dx}\frac{dr'}{dz}\right)$$
$$+ \left(\frac{dr}{dz} + \frac{dr'}{dz}\right)\left(\frac{dr}{dx}\frac{dr'}{dy} - \frac{dr}{dy}\frac{dr'}{dx}\right) = 0,$$

which being an identical equation, the first part of the proposition is true.

The second follows very simply, for let TPT' be the intersection of the tangent plane with the plane SPH; then if ds be an element of the line TPT',

$$\cos SPT = \frac{dr}{dx}\frac{dx}{ds} + \frac{dr}{dy}\frac{dy}{ds} + \frac{dr}{dz}\frac{dz}{ds},$$

and $$\cos HPT = \frac{dr'}{dx}\frac{dx}{ds} + \frac{dr'}{dy}\frac{dy}{ds} + \frac{dr'}{dz}\frac{dz}{ds};$$

therefore, by the fundamental equation (A),

$$\cos SPT + \cos HPT = 0,$$

or $$SPT = 180^\circ - HPT = HPT',$$

and hence $$SPG = HPG,$$

which is the second part of the proposition.

H. G.

Cambridge, Sept. 29, 1847.

[The following geometrical proof, although not new, may be added, in connection with the preceding.

With S and H as foci, and SH as axis of revolution, describe a prolate spheroid, touching the reflecting surface in P. Then SPH is the course of the incident and reflected ray, since the plane SPH, passing through the axis of the spheroid, is perpendicular to the tangent plane at P, and SP, PH, by the known property of the ellipse, make equal angles with the normal PG. Now the value of $SQ + QH$, for any point Q, without the spheroid, is, as follows from another well-known property of the ellipse, greater than $SP + PH$. Hence if the spheroid is touched externally by the reflecting surface, the actual course of the incident and reflected ray is less than if the point of incidence on the surface were in any other position Q, in the neighbourhood of P. We see also that, in general, the point of incidence, P, on the surface is determined by the maximum or minimum condition; although in some cases $SP + PH$ may be actually a maximum, and in others neutral.]

END OF VOL. II.

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